Journal of Statistical Physics, Vol. 42, Nos. 5/6, 1986

Random Surfaces in Statistical Mechanics: Roughening, Rounding, Wetting,...

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Received October 16, 1985

In this paper we study several problems in statistical mechanics involving systems of fluctuating extended objects, such as interacting steps and domain walls. We reconsider the roughening transition and relate it to the free energy of a gas of steps and to the rounding of facets in the equilibrium shape of crystals, defined via the Wulff construction. Using an idealized description due to Fisher and Fisher we analyze the commensurate–incommensurate transition in uniaxial systems in terms of a gas of interacting domain walls. We also study the interactions between a domain wall and a rigid wall and between two interfaces, a problem which is central for the understanding of wetting. Among our results are a quantitative analysis of entropic repulsion between extended objects and a calculation of deviations from mean-field theory in the commensurate–incommensurate transition in dimension $2 \leqslant d \leqslant 3$.

KEY WORDS: Random surfaces; roughening; commensurate–incommensurate transition; wetting.

1. INTRODUCTION

From the growth of crystals to confinement in gauge theories, there is a large variety of physical phenomena connected with the statistical mechanics of random surfaces. In recent years, considerable progress has been achieved in constructing a mathematically rigorous and physically interesting theory of one random surface. The main goal of this paper is to undertake a mathematical analysis of some of the problems dealing with several interacting random surfaces.

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0022-4715/86/0300-0743$05.00/0 © 1986 Plenum Publishing Corporation

The model of random surfaces which is best understood is the solid-on-solid (S.O.S.) model which provides an approximate description of an interface separating two phases in equilibrium, such as a liquid and a vapor phase, or a positively and a negatively magnetized phase in the Ising model. One considers a lattice surface which is the graph of a function, $\phi = (\phi_x)_{x \in \mathbb{Z}^d}$, defined over a parameter space $\mathbb{Z}^d$ (we shall use the word "surface" for all d), and taking values in $\mathbb{Z}$. It serves to model an interface in a lattice system, such as an Ising model. If the surface models a continuous, e.g., liquid–vapor, interface then ϕ is real valued. The Hamiltonian is proportional to the total area of the surface. Suppose that we pin the surface at a given height on the boundary of a box, Λ. One of the main problems is then: Does the average height of the surface inside Λ remain bounded in the thermodynamic limit, $|\Lambda| \to \infty$? (Here and in the following $|\Lambda| \to \infty$ means that $\Lambda \nearrow \mathbb{Z}^d$, through a sequence of increasing subsets.)

For the S.O.S. model, with ϕ_x taking values in $\mathbb{Z}$, we know that a phase transition, called *roughening transition*, takes place only in dimension $d = 2$; At low temperatures the surface is localized (bounded fluctuations),[1-3] while it is delocalized at high temperatures.[4] For $d = 1$ or $d \geqslant 3$, the situation is not as interesting. In the first case, the interface is delocalized at all temperatures $T > 0$,[5] while it is always localized in the second case.[6,7] If we let ϕ_x vary continuously over the reals, then the temperature can be scaled away, and we know that, for $d = 1, 2$, the surface is *delocalized*,[8] while it is *localized* when $d \geqslant 3$.[7]

Several questions remain open: Which of these results, proven for the S.O.S. model, extend to the Ising, or the liquid–vapour, interface? What can be proven about critical phenomena associated with the roughening transition?

Another problem with the roughening transition is that it is hard to detect directly. For $d = 2$, the divergence of the height of the surface in $|\Lambda|$ is proportional to $(\log |\Lambda|)^{1/2}$ and is therefore a *microscopic effect*, *even for a macroscopic* Λ. This problem will be the subject of Section 2. The standard answer (for a review see Refs. 9 and 10) is that the roughening transition reveals itself in the *rounding of facets in the equilibrium shape* of a (crystalline) droplet of one phase, surrounded by the other phase. In order to demonstrate this claim, one appeals to the Wulff construction:[11] Consider the surface tensoin associated with an interface tilted by an angle θ with respect to a lattice axis. Then a cusp in the graph of this surface tension, as a function of θ, implies the presence of a facet in the equilibrium crystal shape. The (one-sided) derivative of the surface tension with respect to θ at $\theta = 0$, and therefore the strength of the cusp, is assumed to be given by the free energy associated with the introduction of a step of height 1 in the interface (i.e., by the *step-free energy*). This is reasonable, since, as we explain in more detail in Section 2, the surface tension of an interface, tilted by an angle θ, can be identified approximately with the free energy of a gas of steps (of height 1) at a density proportional to θ, for θ small. Therefore we expect that the presence of a cusp in the surface tension, at $\theta = 0$, will be equivalent to a nonzero step-free energy, and that the roughening transition should be reflected in the vanishing of the step-free energy.[4] In Section 2, we present a result which makes this picture somewhat more rigorous: For the Ising model and the S.O.S. model, we prove a correlation inequality which says that a nonzero step-free energy implies a cusp at $\theta = 0$ in the graph of the surface tension associated with an interface tilted by an angle θ. We pose, but do not solve, the problem of constructing statistical mechanical methods (e.g., some sort of Mayer expansion) for a gas of extended objects, such as the steps appearing in a tilted interface, or the vortex lines in a superconductor, etc. (See Ref. 20 for some results.)

In Section 3, we make some progress in this direction by analyzing a gas of random surfaces (or random lines), constrained to lie one above another. This problem does not only occur in the study of tilted interfaces, but also in the *commensurate–incommensurate transition*, where the surfaces separate different commensurate regions.[12] Following Ref. 12, we simplify the problem by considering every other interface as rigid and flat (we call it a "wall"). Thus we are led to study one random surface constrained to fluctuate between two walls, put up at height $-l$ and $+l$.

To be specific, we consider the model with the following partition function:

$$Z_\Lambda = \int_{-l}^{+l} \cdots \int \prod_{x \in \Lambda} d\phi_x \exp\left[- \sum_{\langle xy \rangle \subset \Lambda} (\phi_x - \phi_y)^2 \right], \qquad \Lambda \subset \mathbb{Z}^d$$

where the integral is replaced by a sum in case we consider an interface on a lattice.

Our first question is: What is the effect of the walls on the free energy, $\psi^0(l)$, of the system, or more precisely, at what rate does $\psi^0(l)$ approach its limit, $\psi^0(\infty)$, as $l \to \infty$? Moreover, since the correlation length, ξ, is infinite, for $l = \infty$, one may ask: How does $\xi(l)$ diverge when $l \to \infty$? The answer to these questions turns out to provide an example of some critical phenomena, where one can rigorously control a deviation from mean-field theory. Indeed, McBryan and Spencer[13] have shown that, for all dimensions, a mean-field bound holds, namely,

$$\xi(l) \leqslant \exp(cl^2)$$

[4] For a rigorous result essentially supporting this claim see Ref. 4.

We prove that this bound is qualitatively correct, for $d \geqslant 3$, but that there are corrections, for $d = 1, 2$. More precisely

$$\xi(l) \cong cl^2, \qquad \text{if} \quad d=1$$
$$\exp(c_1 l) \leqslant \xi(l) \leqslant \exp(c_2 l), \qquad \text{if} \quad d=2$$

The result for $d = 1$ follows easily from the transfer-matrix formalism, (see Ref. 12 and Appendix 2), but the result for $d = 2$ is more delicate (see Theorem 3.1). Using the information on $\xi(l)$ we get

$$|\psi^0(l) - \psi^0(\infty)| \cong \begin{cases} l^{-2}, & d=1 \\ \exp(-cl), & d=2 \\ \exp(-cl^2), & d \geqslant 2 \end{cases}$$

(see Theorem 3.1). These results permit us to give a rigorous mathematical basis to some of the *Ansätze* in an analysis of the commensurate–incommensurate transition in Ref. 12.

In Section 4, we discuss the interaction of one random surface with one fixed wall, i.e., the surface is constrained to fluctuate above a rigid wall. The surface separates a droplet of a B phase put on the wall from a surrounding A phase. When phases A and B are in equilibrium, one wants to know whether phase B wets the wall or not. In our model, we constrain the surface to stick to the wall on the boundary of a box Λ, and we ask whether the height of the surface above the wall near the center of the wall diverges in the thermodynamic limit, $|\Lambda| \to \infty$, in which case we say that the *B phase wets the wall*. There is an extensive literature on the wetting transition; a transition which can be of first or of higher order. For reviews see Refs. 14–17. However, rigorous results mostly concern two-dimensional models, i.e., one-dimensional surfaces, or lines.[18,19] We investigate these models in higher dimensions; in particular, we analyze in detail the phenomenon of *entropic repulsion*: Consider an S.O.S. surface which, in the absence of a wall, would be almost flat at low temperatures or in dimension $d > 2$. We show that, under the same conditions, the height of this surface above the wall diverges. This is an entropic effect. The surface has more freedom to fluctuate if it is far from the wall than if it is close to it. We obtain precise estimates on the rate of divergence of this height, as a function of the size of the box, in different dimensions and for a variety of models.

Of course we can add, in our models, a potential attracting the surface towards the wall. Then, at low enough temperatures and for sufficiently strong attraction, the surface sticks to the wall. As we raise the temperature, we reach the wetting transition, and the surface detaches itself from the wall. We shall not discuss this transition in detail here. It can be analyzed[20] for weak potentials using the Pirogov–Sinaï theory.[21] For, if the potentials are weak, the transition occurs at low temperatures and is of first order.

We also study a continuum interface interacting with a wall, ($\phi_x \in \mathbb{R}_+$). We expect this model to be adequate when the wetting transition is of second order. Again, we show that the interface is repelled to infinity by the wall, *even* in situations ($d \geqslant 3$) where it would be essentially flat if the wall were absent.

In the last section (Section 5) we briefly discuss another aspect of wetting.[15] Suppose that three phases, A, B, and C, coexist and that the surface tensions between the different phases favor the introduction of a layer of phase C at an interface separating the A and the B phase. Will this layer become macroscopic, in the thermodynamic limit? This problem involves two random surfaces, one being the A–C interface, the other being the C–B interface, constrained to lie one above the other. Although we have fewer results, we expect this situation to be quite similar to the one in Section 4, and our arguments partially support this idea.

The problems analyzed in this paper, though concerning different physics, have a certain logical coherence. This justifies treating them in one paper.

In order to make the reading of this paper lighter, we defer all technical proofs to several appendices. The main body of the paper is outlined as follows.

Section 2: (a) Description of the random surface models (S.O.S. and discrete Gaussian), and survey of known results.

(b) Angular dependence of the surface tension, step-free energy, and their relation (via the Wulff construction) to the shape of crystals. Our main results are inequalities (2.8), (2.9).

Section 3: (a) Relation between the commensurate–incommensurate transition and the ϕ-cutoff Gaussian (or S.O.S.) model, following Ref. 12.

(b) Effect of the ϕ cutoff on this model when the surface has continuously varying heights, (McBryan–Spencer model[13]). Estimates on the free energy, the correlation length, and the variance of the one-spin distribution (Theorem 3.1).

(c) Results for the discrete Gaussian model with a ϕ cutoff: Bounds on the free energy (Theorem 3.2) and analysis of the phase diagram (consequence of the Pirogov–Sinaï theory) at low temperatures, and temperature-independent bounds on the susceptibility; see (3.19).

Section 4: (a) The wetting transition: Models, exact results and correlation inequalities; see (4.6).

(b) Entropic repulsion between a surface and a wall. Application of the Pirogov–Sinaï theory (Theorem 4.1).

(c) Phase diagram for weak attractive potentials. For stronger potentials, we introduce a model of continuous surfaces and analyze the entropic repulsion in this context; see inequality (4.14).

Section 5: Wetting of an interface between two phases by a third phase. Models of two random surfaces, and comparison with the results of Section 4.

Appendix 1: Proof of correlation inequality (2.8).

Appendix 2: Proofs of all the results concerning the ϕ-cutoff continuum Gaussian model (Theorem 3.1 in Section 3.2), and inequality (4.13) in Section 4.3).

Appendix 3: Proofs of all the results concerning the ϕ-cutoff discrete Gaussian model (Theorem 3.2 in Section 3.3 and Theorem 4.1 in Section 4.2).

2. ONE RANDOM SURFACE

2.1. Surface Models and Their Phase Diagrams

The most commonly studied models of interfaces are probably the Ising and the solid-on-solid (S.O.S.) interface. The d-dimensional Ising interface is defined as follows: We take a cylinder $\Lambda = [-L, L]^d \times [-M, M] \subseteq \mathbb{Z}^{d+1}$ and we impose the following $(\pm)$ boundary conditions:

$$\sigma_x = -1 \text{ if } x \notin \Lambda, \qquad x_{d+1} < 0$$
$$\sigma_x = +1 \text{ if } x \notin \Lambda, \qquad x_{d+1} \geqslant 0$$

We have an infinite flat contour outside Λ separating + and − spins (between $x_{d+1} = 0$ and $x_{d+1} = -1$). Every configuration in Λ exhibits a (connected) extension of that contour which we call the interface. The shape of this interface may be quite complicated, and its statistical weight, obtained by summing the Boltzmann factor over all configurations having a given interface, is not simple either. However, this model can be analyzed completely at low temperatures.[3,22–24] In order to investigate what happens at higher temperatures and also because it is an interesting model of crystal growth, one often considers the simpler S.O.S. or discrete Gaussian models: at each site x of a d-dimensional lattice, we attach a variable $\phi_x \in \mathbb{Z}$. The Hamiltonian in a finite box $\Lambda \subset \mathbb{Z}^d$ is

$$H_{\Lambda,\alpha}(\phi) = \sum_{\langle xy \rangle \cap \Lambda \neq \varnothing} |\phi_x - \phi_y|^{\alpha} \tag{2.1}$$

where the sum extends over nearest-neighbor pairs and α is usually taken to be 1 (S.O.S. model) or 2 (discrete Gaussian (D.G.) model). We choose as boundary conditions $\phi_x = 0$, $x \notin \Lambda$, in (2.1). It is easy to see that this model mimics the Ising interface, ϕ_x being the height (in the x_{d+1} direction) of the interface and $\phi_x = 0$ corresponding to an interface between $x_{d+1} = -1$ and $x_{d+1} = 0$. That $\phi_x \in \mathbb{Z}$ means that we have taken the $M \to \infty$ limit in the Ising model. Actually the S.O.S. model ($\alpha = 1$) can be obtained as a limit of anisotropic Ising models, where we let the coupling constant in the direction x_{d+1}, perpendicular to the interface, tend to infinity. The D.G. model ($\alpha = 2$) is quite similar to the S.O.S. model, although it does not always have the same behavior (see Section 3, Remarks after Theorem 3.2). It is a convenient model to analyze mathematically, because the related continuum model (where $\phi \in \mathbb{Z}$ is replaced by integration over $\phi \in \mathbb{R}$) is exactly solvable: It is the Gaussian with mean 0 and covariance $(-\Delta)^{-1}$.

In both models, the partition function is

$$Z_{\Lambda,\alpha,\beta} = \sum_{\substack{\phi_x \in \mathbb{Z} \\ x \in \Lambda}} \exp[-\beta H_{\Lambda,\alpha}(\phi)] \tag{2.2}$$

Interesting quantities to analyze in these models are: The moments of the height distribution, $\langle |\phi_x| \rangle_\Lambda$ or $\langle \phi_x^2 \rangle_\Lambda$, and the correlation between heights at different sites: we may consider the behavior of $\langle \phi_x \phi_y \rangle$ for $|x-y|$ large (if $\langle \phi_x^2 \rangle$ is finite), or else study $\langle (\phi_x - \phi_y)^2 \rangle$ (if $\langle \phi_x^2 \rangle$ is infinite).

It turns out that the rigorous analysis of these models is quite complete, with the following results:

(i) For $d = 1$, $\langle |\phi_0| \rangle_\Lambda \simeq |\Lambda|^{1/2}$; moreover, $\langle (\phi_x - \phi_y)^2 \rangle \simeq |x-y|$. This holds for all temperatures.[5]

(ii) For $d = 2$, there is a transition, called the roughening transition:

(a) For β large, $\langle |\phi_0| \rangle_\Lambda$ is uniformly bounded in $|\Lambda|$. More precisely,

$$P_{\Lambda,\alpha,\beta}(|\phi_0| \geqslant n) \leqslant \exp(-cn^{\alpha}), \qquad \forall n \in \mathbb{N}$$

where $P_{\Lambda,\alpha,\beta}$ is the Gibbs probability distribution associated with (2.1). Finally,

$$\langle \phi_x \phi_y \rangle \leqslant \exp(-m\,|x-y|)$$

with c, $m \cong 0(\beta)$, as $\beta \to \infty$.[1,2]

(b) However, for β small, $\langle |\phi_0| \rangle_\Lambda \cong (\log |\Lambda|)^{1/2}$ and

$$\langle (\phi_x - \phi_y)^2 \rangle \cong \log |x-y| \qquad \text{(Ref. 4)}$$

(iii) For $d=3$, there is no phase transition. Indeed, for all β, $\langle |\phi_0| \rangle_\Lambda$ is uniformly bounded in Λ, and $\langle \phi_x \phi_y \rangle \leqslant \exp(-m\,|x-y|)$ with $m(\beta) \cong 0(\beta)$, as $\beta \to \infty$, and $m(\beta) \cong \exp(-\text{const}\,\beta)$, as $\beta \to 0$.[6] (This last point has been proven only for $\alpha = 2$ but holds probably also for $\alpha = 1$.)

The results can be summarized as follows: Take $\alpha = 2$ in (2.1) and replace the sum over $\phi \in \mathbb{Z}$ in (2.2) by an integral over $\phi \in \mathbb{R}$ with the Lebesgue measure. Then we obtain the massless Gaussian model whose solution is well known.[25] After proper rescaling, this continuum model is the $\beta \to 0$ limit of the D.G. model. Now, for $d=1$, the discrete model behaves like the continuum model, for all β (in one-dimensional systems the high temperature phase extends over all $T > 0$). For $d=2$, this is true only for small β, since for low temperatures the moments of $|\phi|$ are bounded, and the discreteness of the model generates a mass. The transition can be described as an "enhancement of symmetry" in the sense that, for β small enough, the continuum ($\beta = 0$) limit governs the behavior of the discrete model, while, for β large, the discreteness of the model is relevant. For $d=3$, the continuum limit never governs the behavior of the discrete model, and an arbitrarily small β generates a mass.

The preceding analysis can be extended to the Ising interface only for $d+1=2$,[23,24] and for $d+1 \geqslant 3$ at low enough temperatures.[3,22]

2.2. Surface Tension and the Shape of Crystals

We have seen that the most interesting dimension for "surface" phenomena is $d=2$, where a roughening transition occurs. However, from a physical point of view, the divergence of $\langle |\phi| \rangle_\Lambda$ is so small ($\cong |\log \Lambda|^{1/2}$) that it is hard to observe it experimentally.[5] It can nevertheless be observed

[5] The amplitude of fluctuations of the Pacific Ocean, ignoring any water waves, but after roughening has taken place is microscopically small. (We thank M. Berry for pointing this out to us.)

indirectly via the study of equilibrium crystal shapes.[9,10] According to the Wulff construction,[11] the equilibrium shape of a crystal is given by

$$W = \{\mathbf{x}\colon \mathbf{x} \cdot \mathbf{n} \leqslant \tau_\beta(\mathbf{n})\}$$

where $\mathbf{n}$ is a unit vector normal to the crystalline surface, and $\tau_\beta(\mathbf{n})$ is the surface tension of an interface perpendicular to $\mathbf{n}$ (see Fig. 1). In order to define this surface tension more precisely, we introduce the following boundary conditions in (2.1):

Let $\Lambda = [-L, L]^d$ be a box of side $2L+1$ (and let $d=2$ for simplicity). Let

$$\begin{aligned} \phi(-L, m) &= 0, \qquad m = -L,..., L \\ \phi(L, m) &= \kappa, \qquad m = -L,..., L \end{aligned} \tag{2.3}$$

and let us delete from (2.1) the terms $|\phi_x - \phi_y|^\alpha$ for $x = (m, \pm L) \in \Lambda$, $y = (m, \pm(L+1)) \notin \Lambda$, $m = -L,..., L$. Thus we have fixed boundary conditions at different heights on opposite sides of the box, perpendicular to the first

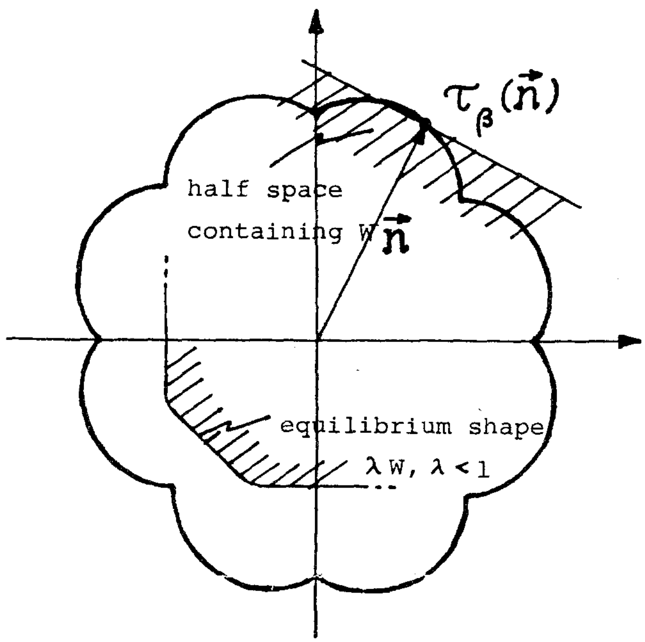


Fig. 1. Polar plot of surface tension, $\tau_\beta(\mathbf{n})$, and equilibrium shape λW, for $\lambda < 1$.

axis, and Neumann boundary conditions along the second axis. Actually, our results also hold for fixed boundary conditions (b.c.):

$$\begin{aligned} \phi(m, \pm L) &= 0, \qquad m = -L,..., 0 \\ \phi(m, \pm L) &= \kappa, \qquad m = 1,..., L \end{aligned} \tag{2.4}$$

Similar b.c. can be defined for the Ising interface (see Appendix 1).

Let $Z_{A,\beta}(\kappa)$ denote the partition function of the S.O.S. model ($\alpha = 1$) with these $(0, \kappa)$-boundary conditions. [Notice that for $\kappa = 0$ it does not reduce to (2.2) if we impose Neumann b.c., while it does if we choose the b.c. (2.4).] The quantity

$$\tau_\beta = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{(2L+1)^2} \log Z_{A,\beta}(\kappa = 0) \tag{2.5}$$

is the *free energy* of the S.O.S. model and is the analog of the *surface tension* in the Ising model. Next, we define $\tau_\beta(\theta) = \tau_\beta(\mathbf{n})$ (where $\mathbf{n}$ and θ are as in Fig. 1) by

$$\begin{aligned} \tau_\beta(\theta) &= -\frac{1}{\beta} \lim_{L \to \infty} \frac{\cos\theta}{(2L+1)^2} \log Z_{A,\beta}[(2L+1)\tan\theta] \\ \tau_\beta(0) &= \tau_\beta \end{aligned} \tag{2.6}$$

and the *step-free energy*,

$$\tau_\beta^{\text{step}} = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L+1} \log[Z_{A,\beta}(1)/Z_{A,\beta}] \tag{2.7}$$

The importance of τ_β^{step} comes from the fact that it equals, formally at least, $\lim_{\theta \to 0}(d/d\theta)\,\tau_\beta(\theta)$. In the Wulff construction, this quantity plays a central role: Since $\tau(\theta)$ is even in θ, its derivative with respect to θ should vanish when θ approaches zero, unless $\tau(\theta)$ has a cusp at $\theta = 0$. However, it is easy to see that a cusp at $\theta = 0$ for $\tau_\beta(\theta)$ is equivalent, in the Wulff plot, to a flat facet in the crystal shape (the size of which is proportional to the strength, $|(d/d\theta)\,\tau_\beta(\theta)\,|_{\theta=0}|$, of the cusp). Therefore, it seems natural to characterize the roughening transition in terms of $(d/d\theta)\,\tau_\beta(\theta)\,|_{\theta=0}$ or, if one may identify this quantity with τ_β^{step}, to define $\beta_{\text{roughening}}$ as

$$\beta_R = \sup\{\beta \mid \tau_\beta^{\text{step}} = 0\}$$

The following is known about τ_β^{step}:

(i) For $d = 1$, $\tau_\beta^{\text{step}} = 0$, for all β, and $\tau_\beta(\theta) \cong c\theta^2$, as $\theta \to 0$.[26]

(ii) For $d=2$, we have a phase transition:

(a′): For β large, $\tau_\beta^{\text{step}}>0$.

(b′): For β small, $\tau_\beta^{\text{step}}=0$.[4] [Compare to (a) and (b) above.]

(iii) For $d=3$, τ_β^{step}, which is dual to the string tension of the $U(1)$ lattice gauge theory (with the Villain action) is strictly positive, for all β.[6]

Thus, the phase diagram, defined in terms of τ_β^{step}, looks similar to the one defined in terms of $\langle|\phi_0|\rangle$. In fact, we expect both descriptions to be equivalent, but it is not proven rigorously that the transition temperatures are identical, for $d=2$.

In Appendix 1, we prove the inequality

$$\tau_\beta(\theta)-\tau_\beta(0)\geqslant|\sin\theta|\;\tau_\beta^{\text{step}} \tag{2.8}$$

for all β, $0\leqslant\theta\leqslant\pi/2$, and $\alpha=1$ in (2.1). This inequality holds not only for the S.O.S., but also for the Ising model (see Appendix 1) and implies that

$$\lim_{\theta\downarrow 0}\frac{d}{d\theta}\tau_\beta(\theta)\geqslant\tau_\beta^{\text{step}} \tag{2.9}$$

This means, as we explained above, that if $\tau_\beta^{\text{step}}>0$ then the equilibrium crystal shape obtained via the Wulff construction contains a flat facet. Of course, one expects equality to hold in (2.8), but we cannot prove this at present.

The following provides a useful representation of $\tau_\beta(\theta)$ and of τ_β^{step}: The partition function $Z_{\Lambda,\beta}(1)$ entering in the definition of the step-free energy (2.7), is, as the name indicates, a sum over surfaces containing (at least) one long step of height 1. This step is a (random) line transversal to the first coordinate axis. In $Z_{\Lambda,\beta}(\kappa)$ we have κ such steps, and thus $\tau_\beta(\theta)$ can be viewed as the free energy of a gas of random lines transversal to the first coordinate axis, the density of these lines being $\tan\theta$. Clearly, τ_β^{step} is the free energy of one isolated line. The equality in (2.8) can be interpreted as the first term in a low-density expansion (density $=\tan\theta\cong\theta$). However, the statistical mechanics of a gas of extended objects (such as lines) remains to be constructed, and therefore it is not straightforward to justify such a low-density expansion. Notice that surfaces contributing to the sum $Z_{\Lambda,\beta}(\kappa)$ contain, in addition to the κ random lines, all kinds of defects that generate effective interactions between the lines. One can hope to control these interactions at low temperatures (see, e.g., Refs. 3 and 23 where similar effective interactions are studied) and to show that they are short ranged. However, even if this is done, a complete low-density expansion of $\tau(\theta)$ does not appear to be easy. Moreover, the existence of a roughening transition at higher temperatures points to a transition in this gas of lines.

3. ONE RANDOM SURFACE BETWEEN TWO WALLS

3.1. Commensurate–Incommensurate Transition in Uniaxial Systems

In this section, we study the uniaxial commensurate–incommensurate transition along the lines laid out by Fisher and Fisher.[12] We manage to supply proofs of some of the assumptions going into their beautiful arguments. Let us consider, as in Ref. 12, an incommensurate phase, which is close to a transition to uniaxial commensurate order and in which an array of domain walls is formed. These walls separate regions of essentially commensurate order. They form a "one-dimensional" gas of fluctuating surfaces (in three dimensions) or lines (in two dimensions) perpendicular to a *preferred* lattice axis. There are short-range interactions and hard-core exclusion between different domain walls. For $d=2$, the situation is quite similar to that found in the gas of steps on surfaces (tilted by an angle θ), described at the end of Section 2.

A simplified model (neglecting short-range interactions other than hard-core) of these interacting lines or surfaces is the following one: Let Λ be a cubic box of side $2L+1$ in a d-dimensional lattice, $|\Lambda|=(2L+1)^d$. If the density of the walls is ρ, we introduce, at each site $x\in\Lambda$, $n(L)\equiv[\rho(2L+1)]$ variables, $\phi_x^1,..., \phi_x^{n(L)}$, with values in $\mathbb{Z}$ or $\mathbb{R}$. (Depending on the situation, we shall consider both cases.) The variable ϕ_x^i represents the height of the ith wall above site x. The Hamiltonian is

$$H_{\Lambda,\alpha}^n(\phi)=\sum_{i=1}^{n(L)} H_{\Lambda,\alpha}(\phi^i), \qquad \alpha=1 \text{ or } 2 \tag{3.1}$$

$H_{\Lambda,\alpha}(\phi^i)$ is given by (2.1), and the partition function is

$$Z_{\Lambda,\beta}(\rho)=\sum_{\boldsymbol{\phi}}^{\leqslant} \exp[-\beta H_{\Lambda,\alpha}^n(\boldsymbol{\phi})] \tag{3.2}$$

where the sum $\sum^{\leqslant}$ extends over all $\boldsymbol{\phi}=(\phi_x^i)_{i=1}^n$ such that

$$-L\leqslant\phi_x^1\leqslant\phi_x^2\leqslant\cdots\leqslant\phi_x^{n(L)}\leqslant L \tag{3.3}$$

For $\phi_x^i\in\mathbb{R}$, the sum in (3.2) is replaced by an integral with Lebesgue measure and the constraint (3.3). This constraint represents the hard-core interaction. The average distance between the walls is $l=(2L+1)/n(L)=\rho^{-1}$.

The free energy of the gas of walls is

$$f_\beta(\rho)=-\frac{1}{\beta}\lim_{L\to\infty}\frac{1}{(2L+1)^{d+1}}\log Z_{\Lambda,\beta}(\rho) \tag{3.4}$$

The free eneergy per domain wall is defined as

$$\psi_\beta(l) = -\frac{1}{\beta} \lim_{L\to\infty} \frac{1}{n(L)(2L+1)^d} \log Z_{A,\beta}(\rho)$$

Since $n(L) = [\rho(2L+1)]$,

$$f_\beta(\rho) = \rho\psi_\beta(l) \tag{3.5}$$

Let τ_β be the free energy of a single wall, in the absence of other walls

$$\tau_\beta = -\frac{1}{\beta} \lim_{L\to\infty} \frac{1}{(2L+1)^d} \log \left\{ \sum_{\substack{|\phi_x| \leqslant L \\ \phi_x \in \mathbb{Z}, x \in A}} \exp[-\beta H_{A,\alpha}(\phi)] \right\} \tag{3.6}$$

We remark that, if we choose to regard (ϕ_x^i) as the location of the ith step in the tilted surfaces of Section 2 then τ_β, as defined here, is the analog of $\tau_\beta{}^{\text{step}}$ in (2.7) (the dimension d here corresponding to $d-1$ in Section 2), and the fact that we do not fix the height ϕ_x along the boundary of A corresponds to Neumann b.c. in (2.7). Moreover, $n(L)$ is the analog of κ in Section 2.2, ρ of $\tan\theta$ in (2.6) and $f_\beta(\rho)$ of $(1/\cos\theta)[\tau_\beta(\theta) - \tau_\beta(0)]$.

We have the following inequality which is the exact analog of (2.8):

$$f_\beta(\rho) \geqslant \rho\tau_\beta \tag{3.7}$$

or

$$\psi_\beta(l) \geqslant \tau_\beta$$

However, in this simple model, (3.7), unlike (2.8), is trivial: by relaxing the constraint that $\phi_x^i \leqslant \phi_x^{i+1}$ in (3.2), we get

$$\sum_{\boldsymbol{\phi}}^{\leqslant} \exp[-\beta H_{A,\alpha}^n(\boldsymbol{\phi})] \leqslant \left\{ \sum_{\substack{|\phi| \leqslant L \\ \phi_x \in \mathbb{Z}, x \in A}} \exp[-\beta H_{A,\alpha}(\phi)] \right\}^n$$

which implies (3.7).

One would really like to know how $\psi_\beta(l) - \psi_\beta(\infty) = \psi_\beta(l) - \tau_\beta$ behaves, as $l = \rho^{-1}$ becomes large. Knowing this would yield the behavior of l, as we vary ψ (ψ acts as a driving potential), and we are particularly interested in the behavior of l when ψ approaches $\psi(\infty)$ (i.e., $l\to\infty$, $\rho\to 0$), the critical point where the commensurate phase sets in.

Computing $\psi_\beta(l)$ or $f_\beta(\rho)$ is not easy. One might try to first fix every other wall, $(\phi_x^i)_{x\in A}$, with i even say, and perform the sum over the walls of odd index. This leads to an effective potential between neighboring walls of even index, and one should then repeat the operation which, we expect, will rapidly converge to $f_\beta(\rho)$, at least for small ρ. Instead of carrying out this construction, we truncate it at the first stage, and we simplify matters further, following Ref. 12, by choosing the walls of even index to be flat and equally spaced ($\phi^i_x = i \cdot l$, i even).

Therefore we have reduced the computation of $f_\beta(\rho)$ to the analysis of one domain wall fluctuating between two planar hard walls separated by a distance $2l$. This system is simple enough that we can analyze it rigorously. Let $\psi^0(l)$ be its free energy,

$$\psi^0_\beta(l) = -\frac{1}{\beta} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log Z^0_\Lambda(l) \tag{3.8}$$

where

$$Z^0_\Lambda(l) = \sum_{\substack{|\phi_x| \leqslant l \\ \phi_x \in \mathbb{Z}, x \in \Lambda}} \exp\left[-\beta \sum_{\langle xy \rangle \subset \Lambda} (\phi_x - \phi_y)^2\right] \tag{3.9}$$

Clearly, $\sum_{\phi_x}$ is replaced by $\int_{-l}^{l} d\phi_x$ if ϕ_x is a continuous variable. For simplicity, we consider the Gaussian case, $\alpha = 2$. We expect, as explained above, that

$$\psi^0_\beta(l) - \psi^0_\beta(\infty) \cong \psi_\beta(l) - \tau_\beta, \qquad \text{as} \quad l \to \infty \tag{3.10}$$

For the left-hand side of (3.10), we can prove, *for continuous* ϕ, that

$$0 \leqslant \psi^0_\beta(l) - \psi^0_\beta(\infty) \cong \begin{cases} cl^{-2}, & d=1 \\ \exp(-cl), & d=2 \\ \exp(-cl^2), & d \geqslant 3 \end{cases} \tag{3.11}$$

with $c = c(\beta)$; see Theorem 3.1.

Throughout this paper we use the shorthand notation: $f(l^\alpha) \cong g(cl^\alpha)$, c a constant, if and only if there are constants c_1, c_2 such that

$$g(c_1 l^\alpha) \leqslant f(l^\alpha) \leqslant g(c_2 l^\alpha) \tag{3.12}$$

Using (3.5), (3.10), (3.11), this result can be reexpressed as

$$f_\beta(\rho) \cong \rho\tau_\beta + \begin{cases} \rho c \rho^{2(\nu-1)/3-\nu}, & \nu < 3 \\ \rho c \exp(-\kappa/\rho), & \nu = 3 \end{cases}$$

where $\nu \equiv d+1$, and we have interpolated in ν. The equilibrium density is obtained by minimizing $f_\beta(\rho)$ with respect to ρ, i.e., by solving

$$\frac{d}{d\rho} f_\beta(\rho) = 0$$

This yields

$$\rho \sim (-\tau_\beta)^{\bar\beta}, \qquad \bar\beta = \frac{3-\nu}{2(\nu-1)}, \qquad \nu < 3$$

and

$$\rho \sim -1/\log(-\tau_\beta), \qquad \text{for} \quad \nu = 3$$

This is the result of Fisher and Fisher.[12] The main point of this section is to provide a proof of (3.11). In Section 3.2 we explain how this is accomplished, along with providing more detailed information on model (3.9), with ϕ continuum. In Section 3.3, we treat the model with ϕ discrete.

3.2. The Continuum Gaussian Model with a Cutoff

We focus our attention on the model defined by (3.9), with $\int_{-l}^{l} d\phi_x$ instead of a sum. We set $\beta = 1$, since it can be absorbed into l by scaling. If we let $l \to \infty$, we obtain the massless Gaussian lattice field whose moments diverge in $d = 1$ and 2. We should ask: How does a cutoff, given by $\chi(|\phi| \leqslant l)$, affect the massless Gaussian field? Or else: What are the critical exponents of this model when the Gaussian critical point $(l \to \infty)$ is approached? In particular, we are interested in the behavior of the correlation length $\xi(l)$,

$$\xi(l)^{-1} = m(l) = \lim_{x \to \infty} \frac{-1}{|x|} \log \langle \phi_0 \phi_x \rangle (l) \tag{3.13}$$

and also of the free energy $\psi(l) - \psi(\infty)$ [defined as in (3.8)], as $l \to \infty$.

A first answer to these questions was provided by McBryan and Spencer,[13] who showed that, for any $l < \infty$, the correlation length $\xi(l)$ is finite. As pointed out by Sokal[27] (see Proposition A1, Appendix 2), this implies that there is a unique Gibbs state, and the notation $\langle (\cdot) \rangle (l)$ in (3.13) referring to the expectation in that state, is unambiguous. The estimate of McBryan and Spencer on $\xi(l)$ is dimension independent and of the form $\xi(l) \leqslant \exp(cl^2)$. As we shall see, this is qualitatively correct for $d \geqslant 3$, but not for $d = 1$ or 2. The $d = 1$ case can be easily settled by using transfer-matrix and random walk ideas (see Ref. 12 and Appendix 2).

Let us briefly review how standard perturbation theory allows us to compute the behavior of $\xi(l)$, as $l \to \infty$, for $d > 1$. We start with a perturbation of the Gaussian which is less singular than $\chi(|\phi| \leqslant l)$, namely, $\exp[-\lambda V(\phi)]$, where $V(\phi)$ is polynomial bounded from below, and λ a small parameter. We keep in mind that

$$\chi(|\phi| \leqslant l) = \lim_{n \to \infty} \exp\left[-\left(\frac{\phi}{l}\right)^{2n} \right] \tag{3.14}$$

and our cutoff perturbation is a limit of smooth perturbations, where the coupling constant $\lambda = l^{-2n}$ tends to zero at the same time as the degree of $V(\phi)$ $(=2n)$ tends to infinity.

In order to compute $m^2(\lambda) = \xi(\lambda)^{-2}$, we consider $S(p) = \sum_x \langle \phi_0 \phi_x \rangle e^{ipx}$, the Fourier transform of the two-point function. One expects $S(p)$ to be approximately given by

$$S(p) \cong \frac{1}{p^2 + m^2(\lambda)} + \text{higher mass terms}$$

and therefore,

$$S^{-1}(p=0) \cong m^2(\lambda)$$

We compute

$$\begin{aligned}\frac{d}{d\lambda} m^2(\lambda) &\cong \frac{d}{d\lambda} [S^{-1}(p=0)] \\ &= \lim_{p \to 0} S^{-2}(p) \left[\sum_{x,y} \langle \phi_0 \phi_x V(\phi_y) \rangle - \langle \phi_0 \phi_x \rangle \langle V(\phi_y) \rangle \right] \end{aligned} \tag{3.15}$$

Now, we approximate the expectation $\langle (\cdot) \rangle$ on the right-hand side of (3.15) by the Gaussian expectation with mass $m^2(\lambda)$, $\langle (\cdot) \rangle (m(\lambda))$, which is expected to be the best effective Gaussian approximation. Then, using Wick's theorem, we obtain

$$\frac{d}{d\lambda} m^2(\lambda) \cong \langle V''(\phi) \rangle (m(\lambda))$$

or, since $m^2(\lambda = 0) = 0$,

$$m^2(\lambda) \cong \lambda \langle V''(\phi) \rangle (m(\lambda)) \tag{3.16}$$

For the cutoff perturbation, $\prod_x \chi(|\phi_x| \leqslant l)$, we find, by the same reasoning,

$$\begin{aligned}\frac{dm^2(l)}{dl} &= S^{-2}(0) \left\{ \sum_y \sum_x \langle \phi_0 \phi_x ; [\delta(\phi_y - l) + \delta(\phi_y + l)] \rangle \right\} \\ &\cong 2S^{-2}(0) \left\{ \sum_y \sum_x \langle \phi_0 \phi_x ; \delta(\phi_y - l) \rangle (m(l)) \right\}\end{aligned}$$

The expectation on the right-hand side being Gaussian it can be evaluated explicitly (integrations by part), and we find to leading order

$$\frac{d}{dl} m^2(l) \cong cl^2 \langle \delta(|\phi| = l) \rangle (m(l))$$

Neglecting powers of l (which turn out to be negligible for $d>1$) and noting that $m(l=\infty)=0$, we obtain the equation

$$m^2(l) \cong \tilde{c}\langle\delta(|\phi|=l)\rangle(m(l)), \qquad \text{for some } \tilde{c}<\infty \tag{3.17}$$

For $d \geqslant 3$, we set $m(l)=0$, on the right-hand side of (3.17), and we obtain

$$m^2(l) \sim \exp(-\alpha l^2)$$

with

$$\alpha^{-1} \cong 2\langle\phi_0^2\rangle \qquad (m=0)$$

For $d=2$, we have

$$m^2(l) \sim \exp[-\alpha(m)\, l^2]$$

with

$$\alpha(m)^{-1} \cong 2\langle\phi_0^2\rangle(m) = \text{const}\ |\log m|$$

which implies

$$m(l) \sim \exp(-\text{const}\ l)$$

Once we know the behavior of $m(l)$, we can also determine the behavior of other quantities of interest, such as $\psi(l)-\psi(\infty)$, by replacing the cutoff $\chi(|\phi| \leqslant l)$ by an effective Gaussian distribution, $\exp[-m^2(l)\phi^2/2]$, and doing all the calculations explicitly with that effective Gaussian measure. For example, if we want to know how $\langle\phi_0^2\rangle(l)$ behaves, as $l \to \infty$, we find

$$\langle\phi_0^2\rangle(l) \sim l, \qquad \text{for} \quad d=2$$

$$\langle\phi_0^2\rangle(l) < \infty, \qquad \text{for} \quad d=3$$

and

$$0 \leqslant \langle\phi_0^2\rangle(l=\infty) - \langle\phi_0^2\rangle(l) \leqslant \exp(-cl^2)$$

Also,

$$\psi(l)-\psi(\infty) \sim \begin{cases} \exp(-cl), & \text{for} \quad d=2 \\ \exp(-cl^2), & \text{for} \quad d=3 \end{cases}$$

That this heuristic calculation actually gives the correct asymptotic behavior of $\langle\phi_0^2\rangle(l)$ and $\psi(l)$, as $l \to \infty$, is the content of the next theorem.

Theorem 3.1. Using the shorthand notation (3.12), we have, for the model defined in (3.9) and with $\phi \in \mathbb{R}$,

$$(1)\ 0 \leqslant \psi(l) - \psi(\infty) \cong \begin{cases} l^{-2}, & d=1 \\ \exp(-cl), & d=2 \\ \exp(-cl^2), & d \geqslant 3 \end{cases}$$

$$(2)\ m(l) = \xi^{-1}(l) \cong \begin{cases} l^{-2}, & d=1 \\ \exp(-cl), & d=2 \\ \exp(-cl^2), & d \geqslant 3 \end{cases}$$

$$(3)\ \langle \phi_0^2 \rangle (l) \cong \begin{cases} l^2, & d=1 \\ l, & d=2 \end{cases}$$

and $0 \leqslant \langle \phi_0^2 \rangle(\infty) - \langle \phi_0^2 \rangle(l) \leqslant \exp(-cl^2)$, for $d \geqslant 3$.

The proof of Theorem 3.1 is based on the ideas described above. We write

$$\chi(|\phi| \leqslant l) = \exp[-m(l)^2 \phi^2/2]\ \chi(|\phi| \leqslant l) \exp[m(l)^2 \phi^2/2]$$

Then we show, using the approximation (3.14), that the "non-Gaussian" part of the interaction $(\chi(|\phi| \leqslant l) \exp[m(l)^2 \phi^2/2])$ is effectively small for l large. This proof uses only elementary tools, like integration by parts and correlation inequalities. It is nevertheless somewhat technical and is therefore presented in Appendix 2.

3.3. The Discrete Gaussian and S.O.S. Models with a Field Cutoff

We now turn our attention to discrete models and examine the effect of the field cutoff, $|\phi| \leqslant l$, in these models. First of all, we state in Theorem 3.2 some bounds for the free energy of the discrete Gaussian model that are related to those of Theorem 3.1. Then we analyze the phase diagram at low temperatures for the discrete models. (That analysis will be useful in Section 4.) Finally, we establish some bounds on the correlation length and on the susceptibility of the discrete Gaussian model which are relevant at higher temperatures.

Theorem 3.2. For the model defined in (3.9), we have, using (3.12):

$$(1)\ 0 \leqslant \psi_\beta^0(l) - \psi_\beta^0(\infty) \cong l^{-2}, \qquad \text{for} \quad d=1 \text{ and for all } \beta$$

$$(2)\ 0 \leqslant \psi_\beta^0(l) - \psi_\beta^0(\infty) \begin{cases} \cong \exp(-cl^2), & d=2 \text{ and } \beta \text{ large enough} \\ \leqslant \exp(-cl), & d=2, \forall\beta \end{cases}$$

$$(3)\ 0 \leqslant \psi_\beta^0(l) - \psi_\beta^0(\infty) \sim \exp(-cl^2), \qquad d=3, \forall\beta$$

Proof. See Appendix 3.

Remark. For the S.O.S. model [$\alpha=1$ in (2.1)], the results are less complete. For $d=1$, of course, we have the same results as above. For $d \geqslant 2$, we can only carry out a low-temperature analysis, which shows that

$$\psi^0(l) - \psi^0(\infty) \cong \exp(-cl) \tag{3.18}$$

for β large enough and $d \geqslant 2$ (see Appendix 3). We do not have good β-independent bounds for this model. We remark that (3.18) differs from the corresponding behavior in the discrete Gaussian model (l instead of l^2!). However, the difference is not too dramatic: (3.18) and Theorem 3.2 both say that $\psi^0(l) - \psi^0(\infty)$ is exponentially small for large l.

The analysis of the *phase structure* of the cutoff discrete ϕ models is quite interesting and important in itself, and this can be done in detail at low temperatures, using the Pirogov–Sinaï theory.[21] Let $d \geqslant 2$, and β be large enough. First, consider $l=\infty$. Then, by a standard Peierls argument,[1,2] one shows that

$$P(|\phi_x| \geqslant \kappa) \leqslant \exp(-c\kappa^\alpha)$$

with α as in (2.1). Here P denotes the probability distribution in the infinite-volume limit.

However, by translation invariance, if we change our b.c. (boundary condition) from $\phi=0$ to $\phi=p$, then we obtain

$$P(|\phi_x - p| \geqslant \kappa) \leqslant \exp(-c\kappa_\alpha)$$

Thus, we can construct infinitely many phases, $\langle(\cdot)\rangle_p$, with $\langle\phi_x\rangle_p = p$, and the set of phases at low temperatures corresponds to the set of ground states, $\phi_x = p$, $\forall x$. It is therefore natural to ask whether there will be $(2l+1)$ phases at low temperatures if we introduce a cutoff $|\phi| \leqslant l$. The answer is *no*; there will be only one phase corresponding to $\phi=0$ or, in other words, if we impose any b.c. $\phi=p$, $|p| \leqslant l$, on the boundary of Λ, we have

$$\langle\phi_x\rangle_{\Lambda,p}(l) \to 0, \qquad \text{as } |\Lambda| \to \infty$$

and

$$P_l(|\phi_x| \geqslant \kappa) \leqslant \exp(-c\beta\kappa^\alpha)$$

in the thermodynamic limit, where c is independent of l and β, (for large enough β).

This fact may seem a little surprising at first sight, but is easy to understand: Although there are $2l+1$ ground states, this does not imply the existence of $2l+1$ pure phases at low temperatures. The correct rule is: The ground states that will produce stable thermodynamic phases at low temperatures are those which admit the largest number of low-energy excitations, thus maximizing an entropy term. We order the excitations around all the ground states according to their energy and count, for each ground state, the number of excitations per unit volume of any given energy. Then we compare two ground states as follows: Ground state A *dominates* ground state B if there is an energy E such that A and B have the same number of excitations for all energies less than E,[6] but A has more excitations of energy E than B; (what happens for energies higher than E does not matter). Then, for low enough temperatures, the stable phases correspond to the set of dominant ground states, i.e., those that dominate all other ground states (and are therefore equivalent to each other, i.e., no ground state in that set has more low-energy excitations than another). The proof of this result[7] (which, in this form, was stated in Ref. 28) is by no means trivial: it follows from a combination of the results of Pirogov and Sinaï[(21)] and of Zahradnik[(29)] (or of Preiss[(30)]).

The application of this principle to our models is easy: of all the ground states, $\phi_x = p$, $|p| \leqslant l$, the one with $p=0$ has more low-energy excitations than any other ground state because we have two "spikes" $\phi_y = +p$, $\phi_y = -p$, with $|p| \leqslant l$, $(\phi_x = 0, |x-y| = 1)$ per unit volume with energy $2d\,|p|^\alpha$. For all the other ground states, some of these excitations will not exist because of the constraint $|\phi_x| \leqslant l$. Another more direct way of explaining this argument goes as follows: Suppose that we have a stable phase corresponding to the ground state $\phi_x = m$; then, in typical configurations, we shall find $\phi_x = m$, for most x, except on a sparse set of excitations. These excitations form a dilute gas, and we can easily compute an approximate free energy for this gas: An excitation, where $\phi_y = m + h$, $\phi_x = m$, $|x-y| = 1$ has an energy $2d\,|h|^\alpha$. Keeping in the partition function only the configurations with these excitations gives

$$-\psi_\Lambda(m) \equiv \log Z_\Lambda(m) \cong |\Lambda| \log \left[\sum_{h=-l-m}^{l-m} \exp(-\beta 2d\,|h|^\alpha) \right]$$

[6] E really measures the energy above a ground state energy (which we have set $= 0$).

[7] Under the additional assumptions of finiteness of the number of periodic ground states and validity of the "Peierls condition"[(21)] which are valid in our models.

Clearly $\psi_\Lambda(m)$, the free energy of this gas of excitations above the ground state $\phi_x = m$ reaches its minimum for $m = 0$. Hence the only stable phase will correspond to $\phi_x = 0$.

Thus, for the discrete models with cutoff $|\phi| \leqslant l$, (with $\alpha = 1$ or 2) we have a unique phase (actually a unique Gibbs state) at low temperatures. We remark that, if we had $2l$ ground states instead of $2l+1$, i.e., if the values of ϕ range over the set $\{l-1/2, l-3/2,..., -l+1/2\}$, we would have two phases, in $d \geqslant 2$ dimensions, corresponding to $\phi = \pm 1/2$, at low temperatures. For the continuous ϕ Gaussian model, in $d \geqslant 3$, and with no cutoff, we can produce, by choosing appropriate b.c., uncountably many phases, $\langle(\cdot)\rangle_\omega$, $\omega \in \mathbb{R}$, with $\langle\phi_x\rangle_\omega = \omega$. However, once we introduce the cutoff $\chi(|\phi| \leqslant l)$, then for any l, there is only one phase, and $\langle\phi\rangle = 0$. The thermodynamic limit becomes *independent* of the b.c. (see Appendix 2, Proposition A1, for a proof).

Now we discuss the correlation length $\xi(l)$ for the discrete Gaussian model. Our first observation is that

$$\langle\phi_0\phi_x\rangle(l) \leqslant \langle\phi_0\phi_x\rangle(l=\infty)$$

by the Griffiths–Nelson inequalities,[31,32] and therefore $\xi(l) < \infty$, uniformly in l, whenever $\xi(l=\infty) < \infty$. This holds in particular for $d = 2$ and β large,[2] and for $d \geqslant 3$ at all values of β.[6] For $d = 1$, it is easy to prove that $\xi(l) \sim l^2$.[12] The most interesting situation is $d = 2$ and β small, i.e., when the $l = \infty$ model is in its rough phase. It is unclear whether the $l < \infty$ model undergoes some kind of transition in this regime. For instance, one observes[8] that, at low temperatures, $\phi_x = 0$, for most x, and therefore there is percolation of the sites where $\phi_x = 0$. At high temperatures, however, one can show that the sites where $\phi_x \neq 0$ percolate (at least, if l is not too small). Does this correspond to some kind of phase transition? All we can say is that, for all d, β, l, the susceptibility

$$\chi(l) = \sum_{x \in \mathbb{Z}^d} \langle\phi_0\phi_x\rangle(l) < \infty \tag{3.19}$$

remains finite, so that the model is never critical [in this sense; we cannot prove, however, that $\xi(l)$ is finite for all β]. In order to prove (3.19), we first apply the van Beijeren–Sylvester inequality[33]

$$\langle\phi_0\phi_x\rangle(l) \leqslant \langle\phi_0\phi_x\rangle(l=\infty, m(l)) \tag{3.20}$$

where on the right-hand side we have a discrete Gaussian model with a

[8] We thank J. T. and L. Chayes for an interesting discussion on this point.

mass $m(l) = \exp(-cl^2)$. Now, if we consider the Fourier transform $S(p) = \sum_{x \in \mathbb{Z}^d} e^{ipx} \langle \phi_0 \phi_x \rangle$, the correlation inequality of Ref. 34 implies

$$S_D(p) \leqslant S_C(p) \tag{3.21}$$

for fixed $m(l)$, where the index D refers to the discrete ($\phi \in \mathbb{Z}$) Gaussian model and C to the continuum ($\phi \in \mathbb{R}$) one. Now, $S_C(p) = [2 \sum_{\alpha=1}^d (1 - \cos p_\alpha) + m^2(l)]^{-1}$ and, combining (3.20) and (3.21), we have

$$\chi(l) \leqslant S_D(p=0) \leqslant m^{-2}(l)$$

4. ONE RANDOM SURFACE AND ONE WALL

4.1. The Wetting Transition

We consider a surface constrained to fluctuate above a rigid wall (see, e.g., Refs. 14–19 and 35–38). This is intended to describe an interface between two coexisting phases, one of which is supported by the wall. Specifically, we study the model with Hamiltonian

$$\begin{gathered} H_\Lambda = \sum_{\langle xy \rangle \cap \Lambda \neq \varnothing} |\phi_x - \phi_y|^\alpha + a \sum_{x \in \Lambda} V(\phi_x) \\ \text{and partition function } Z_\Lambda = \sum_{\phi_x = 0, x \in \Lambda}^{\infty} \exp(-\beta H_\Lambda) \end{gathered} \tag{4.1}$$

We shall focus our attention on the case $a = 0$. We shall however also consider the situation where a potential $V(\phi)$ attracts the surface towards the wall; for example (with $a > 0$)

$$V(\phi) = \begin{cases} -\delta(\phi) & (4.2) \\ -\exp(-\kappa\phi), \qquad \kappa > 0 & (4.3) \end{cases}$$

For a more detailed analysis see also Ref. 20.

One could include, in (4.1), a chemical potential $\mu\phi$ ($\mu > 0$) which would occur naturally if the phase lying on the wall does not coexist with the bulk phase. In this case, one may also consider a repulsive potential [$a < 0$ in (4.1)].[35–38]

This model can be viewed as an approximation to an Ising interface of the following type[18,19,35–38]: Consider a semi-infinite lattice $\mathbb{Z}_+^{d+1} = \{x \in \mathbb{Z}^{d+1} / x_{d+1} \geqslant 0\}$, and fix the b.c. $\sigma_x = -1$, for $x_{d+1} = 0$. This induces a negative magnetic field on the spins σ_x, $x_{d+1} = 1$. If we impose +b.c. ($\sigma_x = +1$) on the other walls of a box $\Lambda = \{x \in \mathbb{Z}^{d+1} \mid 0 < x_{d+1} < M, -L \leqslant x_\alpha \leqslant L, \alpha = 1,..., d\}$, we obtain an interface in Λ separating + and −

spins. An approximation of the shape and the weights of this interface leads to (4.1), with $a=0$. If we weaken the bonds between $x_{d+1}=0$ and $x_{d+1}=1$, by replacing the coupling constant J by SJ, $0<S<1$, then our approximation yields (4.1), with $V(\phi)=-\delta(\phi)$.

We study the following questions: If we impose the boundary conditions $\phi_x=0$, $x\notin \Lambda$, what is the behavior of $\langle\phi_0\rangle_\Lambda$, when $|\Lambda|\to\infty$, as a function of β and a? How does the correlation length within the surface depend on β and a?

In one dimension, the problem is completely solved, using transfer-matrix methods[16,18,19,39]: There exists a temperature $T_W(a)$, for $a>0$, such that

$$\lim_{|\Lambda|\to\infty}\langle\phi\rangle_\Lambda<\infty, \qquad \text{for} \quad T<T_W(a) \tag{4.4}$$

$$\lim_{|\Lambda|\to\infty}\langle\phi\rangle_\Lambda=\infty, \qquad \text{for} \quad T>T_W(a) \tag{4.5}$$

At $T_W(a)$, the wetting transition occurs: For T above $T_W(a)$, a droplet of the phase lying on the wall becomes macroscopic and completely wets the wall (or dries the wall, if the phase on the wall is gaseous[37]). For $d=1$, this wetting transition is second order: The correlation length $\xi=m^{-1}$ defined, for $T<T_W(a)$, by

$$m=-\lim_{x_1\to\infty}\frac{1}{|x_1|}\log(\langle\phi_y\phi_{y+x_1}\rangle-\langle\phi_y\rangle^2)$$

diverges, as $T\uparrow T_W(a)$, like $|T-T_W|^{-2/3}$; $\langle\phi_0\rangle=\lim_{|\Lambda|\to\infty}\langle\phi_0\rangle_\Lambda$ diverges as $|T-T_W|^{-1/3}$.[39]

For $a=0 (d=1)$ and all $T>0$, $\langle\phi_0\rangle_\Lambda\cong|\Lambda|^{1/2}$. Thus $T_W(a=0)=0$. This is an instance of a general inequality (valid for arbitrary d)

$$T_W(a)\leqslant T_R(a) \tag{4.6}$$

where $T_R(a)$, the roughening temperature, corresponds to the transition from (4.4) to (4.5), but for the model where $\phi\in\mathbb{Z}$ in (4.1) [and with $|\phi|$, instead of ϕ, in (4.4), (4.5)]. For $a=0$, this is of course the model discussed in Section 2 and, as we explained there, $T_R(a=0)=0$. Inequality (4.6) is a direct consequence of F.K.G. inequalities[40]:

$$\langle\phi^p\rangle(\phi\geqslant 0)\geqslant\tfrac{1}{2}\langle|\phi|^p\rangle(\phi\in\mathbb{Z}), \qquad \forall p>0 \tag{4.7}$$

because both $F(\phi)=|\phi|^p\,\chi(\phi\geqslant 0)$ and $\chi(\phi\geqslant 0)$ are increasing functions of ϕ, and $|\phi|^p=F(\phi)+F(-\phi)$. Inequality (4.6) and an other version of (4.7) also hold for the corresponding Ising model interface, as one checks by using FKG inequalities.[40]

4.2. One Surface and One Wall, Without an Attractive Potential

Now we turn our attention to $d \geqslant 2$, $a=0$, and β large. We know that $T_R(a=0)>0$, i.e., at low temperatures the interface is essentially flat and $\langle|\phi|\rangle_\Lambda < \infty$ uniformly in Λ (see Section 2.1). However, $T_W(a=0)=0$; indeed, we shall show that, for all d, all β large enough, and $a=0$,

$$\lim_{|\Lambda| \to \infty} \langle \phi \rangle_\Lambda = \infty \tag{4.8}$$

Again, this may be a little surprising but is easy to understand and is essentially the same phenomenon as the one discussed in Section 3.3, namely, that the cutoff ($|\phi| \leqslant l$) models have a unique phase. Indeed, for any l,

$$\langle \phi_x \rangle_\Lambda \geqslant \langle \phi_x \rangle_\Lambda (0 \leqslant \phi \leqslant 2l) \tag{4.9}$$

where in the right-hand side ϕ_x takes the $2l+1$ values $0, 1, \ldots, 2l$. This follows again from F.K.G. inequalities,[40] because ϕ is an increasing function and $\chi(\phi \leqslant 2l)$ a decreasing one. Now, perform the translation $\phi_x = \phi'_x + l$. Then, ϕ'_x has the same distribution as the cutoff models ($|\phi'| \leqslant l$) of Section 3.3. The $\phi=0$ b.c. become $\phi' = -l$. However, since the cutoff models have a unique phase, $\langle \phi'_x \rangle_{\Lambda, -l} \to 0$ as $|\Lambda| \to \infty$, and so, $\langle \phi_x \rangle_\Lambda (0 \leqslant \phi \leqslant 2l) \to_{|\Lambda| \to \infty} l$. The lower bound (4.9), for the ($\phi \in \mathbb{Z}_+$) model, gives $\lim_{|\Lambda| \to \infty} \langle \phi_x \rangle_\Lambda \geqslant l$ for any l, whence (4.8).

This phenomenon is an instance of *entropic repulsion*: The surface (ϕ_x) wants to stay away from the wall, in order to have more freedom to fluctuate. This effect is due to low-energy excitations. If the surface is at a height h above the wall ($\phi=0$) it has more (in fact, twice as many) spikes of energy $2d(h)^\alpha$ than at level $h-1$ (spikes growing "downwards," $\phi_y=0$, $\phi_x=h$, $|x-y|=1$, are allowed at level h, but not at level $h-1$, because of the constraint $\phi \geqslant 0$. Those growing "upwards" are allowed, for both levels).

The following rough calculation gives more quantitative information on the behavior of $\langle \phi_x \rangle_\Lambda$ (see also Ref. 38): Suppose that the interface is at a height $h-1$ near the boundary of a box of side L. If we increase the height from $h-1$ to h, we increase the free energy by an energy term of order βL^{d-1} (formation of a terrace). But, because more elementary excitations are allowed at level h, we also decrease the free energy by an "entropy" term of order $\exp(-\beta 2dh^\alpha)\, L^d$. The balance between both terms is achieved for h of order $[\beta^{-1} \log(L/\beta)]^{1/\alpha}$, $\alpha = 1$ or 2.

These heuristic considerations can be made precise, using the analysis of Pirogov and Sinaï,[21] and yield the following.

Theorem 4.1. For the model defined in (4.1), with $a=0$, $d\geqslant 2$, and β large enough,

$$|\Lambda|^{-1} \sum_{x\in\Lambda} \langle\phi_x\rangle_\Lambda \cong \begin{cases} (C/\beta)\log|\Lambda|, & \text{for} \quad \alpha=1 \\ ((C/\beta)\log|\Lambda|)^{1/2}, & \text{for} \quad \alpha=2 \end{cases}$$

where we used (3.12), and C is independent of β.

In Appendix 3 we give an "elementary" proof of this result. It is based on ideas of Refs. 21, 29, 30, and 41, but our proof is self-contained in the sense that we do not appeal explicitly to their results.

Remarks. (1) In order to convince the reader that this entropic repulsion is due to the spikes growing downwards from the surface to the wall, let us contrast the present situation with the one of the "wedding cake" model[42]:

Take the Hamiltonian (4.1), with $a=0$, as before. We shall describe the surface ϕ_x in terms of *walls* and *ceilings*[3] (see Appendix 3 for a more detailed definition): A ceiling is a connected set where ϕ_x is constant while a wall (similar to a contour in the Ising model) is a connected set of bonds where $\phi_x \neq \phi_y$. The complement of a wall in $\mathbb{Z}^d$ is made of one infinite ceiling, called the exterior ceiling, and one or several finite, interior ceilings. The wedding cake model is defined by the following constraint: A wall can only lift the surface, i.e., the interior ceiling(s) must be higher than the exterior ceiling. One might expect that this constraint would make $\langle\phi_x\rangle$ even larger than in the original model but, at least for the ($\phi\geqslant 0$) model, this is not so: For the wedding cake model, it is fairly easy to show, using a Peierls argument, that

$$\langle\phi_x\rangle < \infty$$

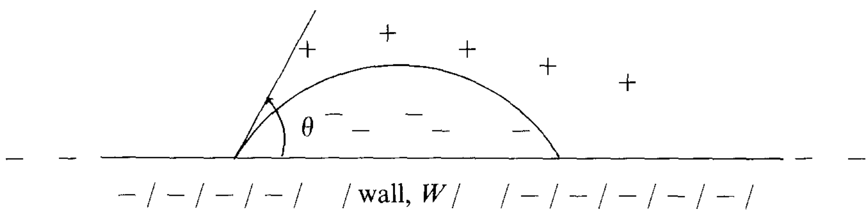
at low temperatures. The surface is similar to the one *without* the ($\phi_x\geqslant 0$) constraint. The reason is that, by construction, there are no spikes growing downwards in the wedding cake model, and such spikes are responsible for the lifting of the surface in the original model. To summarize, in the original model, the surface gets pushed upwards, away from the wall, *only* in order to enhance the growth of "small" downwards fluctuations which increase the entropy of the surface.

(2) For the Ising interface on the semi-infinite lattice, $\mathbb{Z}^d_+$ (see Section 4.1), we expect the same phenomenon to occur (this is also supported by mean-field computations[36]), but a proof would be more technical. Using the inequalities of Ref. 43, this result would imply the uniqueness of the Gibbs state for this model and thus, with these b.c. imposed there should be no (surface) phase transition in the semi-infinite Ising model.

Here we just present a heuristic calculation supporting these claims, but we expect that a rigorous proof would follow from an appropriate extension of Pirogov–Sinaï theory. We impose $-$ b.c. at the bottom wall of a region $\Lambda = [-L, L]^2 \times [0, M] \subset \mathbb{Z}^3$ ($M \to \infty$), and $+$ b.c. at the remaining faces of $\partial\Lambda$. Let $\sigma = \sigma(\beta)$ denote the usual surface tension of the three-dimensional Ising model, and let σ_{+W} denote the surface tension between the $+$ phase and the bottom wall, σ_{-W} the surface tension between the $-$ phase and the bottom wall. Antonov's rule for the shape of a $-$ droplet attached to the bottom wall says that

$$\text{(i)}\quad \sigma \cos\theta = \sigma_{+W} - \sigma_{-W}$$

where θ is the contact angle between the surface of the droplet and the wall.



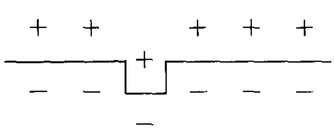
Heuristically, the $-$ phase *wets* the wall W if (i) does *not* admit any real solution for the angle θ, i.e., if

$$\text{(ii)}\quad \sigma < \sigma_{+W} - \sigma_{-W}$$

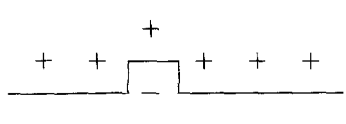
A heuristic proof of (ii) may be obtained easily by calculating the leading contributions to σ, σ_{+W}, and σ_{-W} in a low-temperature expansion. These contributions are

$$\text{(iii)}\quad \sigma \simeq \frac{1}{\beta}(2\beta - 2e^{-4\beta}) = 2 - \frac{2}{\beta}e^{-4\beta}$$

This follows by studying the leading low-energy excitations,

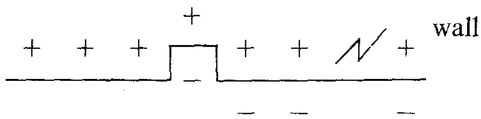


and



$$\text{(iv)}\quad \sigma_{+W} \simeq \frac{1}{\beta}(2\beta - e^{-4\beta})$$

as one sees by considering

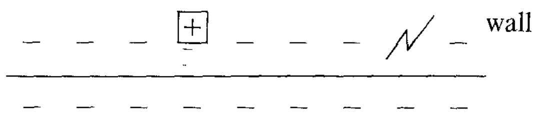


The reflected (downward) excitation is *missing*, because the wall is rigid. (This results in entropic repulsion.)

(v) For σ_{-W}, we find

$$\sigma_{-W} \simeq \frac{1}{\beta}(e^{-6\beta})$$

which is negligible. This is seen from



Hence

$$\begin{aligned}\sigma &\simeq 2 - \frac{2}{\beta} e^{-4\beta} \\ &< 2 - \frac{1}{\beta} e^{-4\beta} - O\left(\frac{1}{\beta} e^{-6\beta}\right) \\ &\simeq \sigma_{+W} - \sigma_{-W}\end{aligned}$$

which is (ii).

We remark that by weakening the bonds adjacent to the wall one can achieve that σ is larger than $\sigma_{+W} - \sigma_{-W}$, and wetting disappears, as one may prove by a Peierls-type argument.

It would be interesting to derive a mathematically precise form of Antonov's rule.

4.3. The Effect of an Attractive Potential

No we consider model (4.1) with $a \neq 0$, $d \geqslant 2$, and V given by (4.2) or (4.3). By using a Peierls argument, it is easy to show that, for any $a > 0$, $\langle \phi_x \rangle < \infty$, for low enough temperatures. What happens when one raises the temperature can be somewhat complicated, depending on V and a. Let us first consider low temperatures, a small, and, to fix the ideas, let $\alpha = 1$ in (A1), and $V = e^{-\kappa\phi}$. We can use again the Pirogov–Sinaï theory, which was extended to these models by Basuev[44] (but with no explicit applications): Then one can exhibit in the (a, $T = \beta^{-1}$) plane, an infinite sequence of lines of first-order phase transitions $a_n(T)$, $n = 1, 2, \ldots, \infty$. These are similar to the layering transitions of Refs. 35–38. On the line $a_n(T)$, we have coexistence between two phases, one with $\phi_x = n - 1$, for most x, and one with $\phi_x = n$, for most x. These lines accumulate towards the $a = 0$ axis and are approximately given by

$$a_n(T) \cong \exp\{-[2d\beta(n+1) - \kappa n]\} \tag{4.10}$$

The complete proof of the occurrence of these lines, for β large, is too long to be included here (see Ref. 20). The approximate equation (4.10) can be derived by using the low-energy excitations along the lines developed in Section 3.3 (see also Ref. 38). We associate a free energy, F_n, to each level $\phi \cong n$ as follows: From (4.1) we infer that there is an energy term, U_n, equal to $-ae^{-\kappa n}$; there is also a loss of entropy (as compared to the level $n = \infty$), due to the spikes growing downwards and forbidden by the constraint $\phi \geqslant 0$, given, approximately, by

$$S(n) - S(\infty) \cong -\exp[-2d\beta(n+1)]$$

Thus

$$F_n = U_n - TS_n \cong -ae^{-\kappa n} + \frac{1}{\beta}\exp[-2d\beta(n+1)]$$

Minimizing F_n with respect to n yields (4.10). All these transitions are first-order, namely, the correlation functions are exponentially decreasing throughout this part of the phase diagram (a, T small).

What happens for larger a's? We expect a similar phase diagram (transition from $\langle\phi_x\rangle < \infty$ to $\langle\phi_x\rangle = \infty$ as one raises the temperature), but the transition may become second order, at least for $d = 2$. Indeed, when $\langle\phi_x\rangle_\Lambda$ diverges, the correlation length, for perturbations within the surface, should not be too much affected by the presence of the wall and should therefore diverge (approximately) when an interface with no wall exhibits massless excitations. In other words, we expect that the wetting transition will be second-order, for values of a such that $T_W(a)$ happens to be larger than $T_R(a = 0)$. (This, however, is not meant to be an exact quantitative statement.) Thus one expects the following.

(i) For $d=1$, the wetting transition is always a second-order transition, since $T_R(a=0)=0$, and this is known to be true.[18,19,39]

(ii) For $d=2$, we have just seen that, for a small, $T_W(a)$ is small [see (4.10)], and the transition is first order. Here, $T_R(a=0)$ is neither zero nor infinite. If we increase a, $T_W(a)$ increases (this holds, by F.K.G. inequalities,[40] for any $V(\phi)$ which is monotone decreasing in ϕ, for $\phi \geqslant 0$). Thus, for appropriate values of a, $T_W(a)$ will be larger than $T_R(a=0)$, and the transition should be second order.

(iii) For $d=3$, since Göpfert and Mack have shown[6] that $T_R(a=0)=\infty$, one expects the transition to be always first order.

Now we would like to say something about the rate of divergence of $\langle\phi_0\rangle_\Lambda$ in $|\Lambda|$, when $T \geqslant T_W(a)$, and a is such that $T_W(a) > T_R(a=0)$. Since we know that, for $T \geqslant T_R$, the discrete ϕ variable of the S.O.S. model should be regarded as effectively continuous,[4] we study a model where ϕ is a continuous variable from the beginning. Also, since we are interested in temperatures larger than $T_W(a)$, where the wall does not bind the surface, we neglect the attractive potential. Thus, we consider the model (4.1) with $a=0$, $\alpha=2$, and ϕ continuous (but with the constraint $\phi \geqslant 0$). It is rather easy to derive, for this model, nontrivial lower bounds on $\langle\phi_x\rangle_\Lambda$ (with b.c. $\phi_x=0$, $x \notin \Lambda$): As in (4.9) we use F.K.G. inequalities[40] to obtain

$$\langle\phi_x\rangle_\Lambda \geqslant \langle\phi_x\rangle_\Lambda(0 \leqslant \phi \leqslant 2l) \tag{4.11}$$

and in the right-hand side, write $\phi=\phi'+l$. Now, the ϕ' variables have the constraint $|\phi'_x| \leqslant l$ and b.c. $\phi'_x=-l$, $x \notin \Lambda$. We know (see Appendix 2) that this cutoff Gaussian model has a unique thermodynamic phase and thus, $\langle\phi'_0\rangle_{\Lambda,-l}(|\phi'| \leqslant l) \to 0$, as $|\Lambda| \to \infty$. Moreover, using G.H.S. inequalities[45,27] [see Eq. (A12) in Appendix 2] we know that, if

$$|\Lambda| = L^d \quad \text{with} \quad L=\xi(l)^{1+\varepsilon}, \qquad \varepsilon > 0 \tag{4.12}$$

and if l is large enough,

$$|\langle\phi'_0\rangle_{\Lambda,-l}(|\phi'| \leqslant l) \leqslant 1 \tag{4.13}$$

Thus by (4.11),

$$\langle\phi_x\rangle_\Lambda \geqslant l-1$$

with l and Λ related by (4.12). Inverting this relation and using the upper bounds on $\xi(l)$ of Theorem 3.1, we get the following lower bounds:

$$\langle\phi_0\rangle_\Lambda \geqslant \begin{cases} c(L)^{1/2}, & \text{for} \quad d=1 \\ c\log L, & \text{for} \quad d=2 \\ c(\log L)^{1/2}, & \text{for} \quad d \geqslant 3 \end{cases} \tag{4.14}$$

Remarks. (1) For $d=3$, since we expect the wetting transition to be always first-order, this lower bound is not expected to be relevant for the wetting problem with discrete ϕ_x's (see, however, Theorem 4.1).

(2) We emphasize that these lower bounds include a nontrivial effect of the constraint $\phi \geqslant 0$ (similar to the effect on discrete ϕ's expressed by Theorem 4.1). Indeed, using F.K.G. inequalities[40] to bound $\langle \phi \rangle_\Lambda$ $(\phi \geqslant 0)$ from below by $\frac{1}{2}\langle |\phi| \rangle_\Lambda$ $(\phi \in \mathbb{R})$ (without the constraint), as in (4.7) would only give

$$\langle \phi \rangle_\Lambda \geqslant \begin{cases} c(\log L)^{1/2}, & \text{for} \quad d=2 \\ \text{const}, & \text{for} \quad d \geqslant 3 \end{cases}$$

5. TWO RANDOM SURFACES

In this section we study a model where, instead of having one surface fluctuating above a fixed wall, we consider two random surfaces interacting through the constraint that one surface lie above the other one (see Ref. 15 and references therein). This situation arises when we analyze three phases in thermal equilibrium, A, B, and C, and a layer of the phase C is developed at the boundary between the A and the B phase, in order to lower the surface tension. Then there are two interfaces, one between A and C, the other one between C and B.

The Blume–Capel model[46,47] provides an explicit example of a model describing this situation: Let $s_x = 0, \pm 1$, $x \in \mathbb{Z}^{d+1}$, and

$$H = \sum_{\langle xy \rangle} (s_x - s_y)^2 - \mu \sum_x s_x^2 \tag{5.1}$$

If we fix $\mu > 0$, we have two phases (with s_x predominantly equal to $+1$ or to -1), at low enough temperatures. Using the Pirogov–Sinaï theory,[21,48] one can prove that, for β large, there exists a curve

$$\mu(\beta) \cong \exp[-2(d+1)\beta]$$

of first-order transitions where three phases coexist (with $s_x \cong +1$, -1, or 0). Let us fix β large and $\mu = \mu(\beta)$, in (5.1). Let $\Lambda = [-L, L]^d \times [-M, M]$ and put $\pm$ b.c. on $\partial\Lambda$: For $x \notin \Lambda$,

$$s_x = +1, \qquad x_{d+1} \geqslant 0$$

$$s_x = -1, \qquad x_{d+1} < 0$$

It is energetically favorable to insert at least one layer of spins $\{s_x = 0\}$ (the C phase) between $\{s_x = +1\}$ and $\{s_x = -1\}$, because, taking into account (5.1), one finds

$$(1-0)^2 + [0-(-1)]^2 = 2 < [1-(-1)]^2 = 4$$

(we neglect terms $\propto \mu$, because μ is small here). The question is then: What will the size of this layer of 0 spins be, as M, $L \to \infty$? Since the 0 phase is in equilibrium with the other two phases, there is nothing, *a priori*, to prevent this layer from becoming macroscopic. This is again a *wetting phenomenon*. Another question is: Fix $\mu > 0$ small enough, and raise the temperature, starting from 0, towards the temperature at which the first-order phase transition occurs. For T strictly below that transition, the layer of the 0 phase is expected to have a finite thickness, because of a bulk effect. How does that thickness grow with the temperature? Does it diverge, as the transition point is approached; i.e., is wetting a continuous or a discontinuous transition in this situation? We do not discuss this last question any further. Instead we introduce the S.O.S. (or D.G.) approximation[15] relevant to this two-surface problem and corresponding to the value $\mu = \mu(\beta)$. Let $\phi_x^1, \phi_x^2 \in \mathbb{Z}$, $x \in \mathbb{Z}^d$. The Hamiltonian is given by

$$\begin{aligned} H_{\Lambda,\alpha}(\phi^1, \phi^2) &= H_{\Lambda,\alpha}(\phi^1) + H_{\Lambda,\alpha}(\phi^2) \\ &\quad + a \sum_{x \in \Lambda} V(\phi_x^1 - \phi_x^2) \end{aligned} \tag{5.2}$$

with $H_{\Lambda,\alpha}$ given by (2.1), $\Lambda = [-L, L]^d$.

The partition function is

$$Z_{\Lambda,\alpha,\beta} = \sum_{\phi_x^1 \geqslant \phi_x^2} \exp[-\beta H_{\Lambda,\alpha}(\phi^1, \phi^2)]$$

and we set $\phi_x^1 = \phi_x^2 = 0$, for $x \notin \Lambda$. If we set $\phi_x^2 = 0$, for all x, we recover the models considered in Section 4. The "potential" $V(\phi_x^1 - \phi_x^2)$ describes a possible attractive force between the two surfaces. One is interested in the qualitative behavior of $\langle \phi_x^1 - \phi_x^2 \rangle_\Lambda$, as $|\Lambda| \to \infty$, as a function of β and the coupling constant a. We expect that all the results of Section 4 remain valid in the present situation. In particular, using the Pirogov–Sinaï theory, one can show[20] that, for $a = 0$,

$$\lim_{\Lambda \to \infty} |\Lambda|^{-1} \left(\sum_{x \in \Lambda} \langle \phi_x^1 - \phi_x^2 \rangle_\Lambda \right) = \infty \tag{5.3}$$

for β large enough. If, however, a is large enough, $\langle \phi_x^1 - \phi_x^2 \rangle_\Lambda$ remains finite, uniformly in Λ.

There is an "*entropic repulsion*" between the two surfaces, just as between a surface and a rigid wall. However, the proofs are more complicated than in the case, where $\phi_x^2 \equiv 0$ (see Ref. 20).

Coming back to the Blume–Capel model, a result like (5.3) strongly suggests that the thickness of the layer of the $\{s_x = 0\}$ phase will grow to infinity, as $|\Lambda| \to \infty$, for $\mu = \mu(\beta)$, or, in other words, that the thermodynamic limit of the Gibbs state with $\pm$ b.c. is the pure, translation-invariant $\{s_x = 0\}$ phase. This is true in all dimensions, d, in contrast to the well-known spin-1/2 Ising magnet ($s_x = \pm 1$), where for $d \geqslant 3$, $\pm$ b.c. lead to a non-translation-invariant state at low temperatures.[3]

Since the proofs of (5.3) and various related results are rather technical and involve methods which we do not wish to introduce in this paper they will be presented in a separate paper. The heuristics of the entropic repulsion between the two surfaces is, however, straightforward and very similar to that presented in Section 4.

ACKNOWLEDGMENTS

We thank E. Brézin and M. E. Fisher for having triggered our interest in some of the problems studied in this paper and for very helpful discussions. We are grateful to D. Abraham, J. T. Chayes, L. Chayes, B. Drrida, D. Disher, R. Kotecky, J. L. Lebowitz, C.-E. Pfister, M. Schick, J. Slawny, A. Sokal, T. Spencer, and M. Zahradnik for their interest in our work and for useful discussions on various parts of this paper. J. B. acknowledges the hospitality of the Physics Department of ETH and of the Mathematics Department of Rutgers University, and J. F. thanks the Institute for Advanced Study for hospitality during periods when part of this work was done.

This work was supported in part by N.S.F. grant No. DMS-8405164 while the author was visiting the Institute for Advanced Study, Princeton, New Jersey 08540.

APPENDIX 1: THE STEP-FREE ENERGY AND THE ANGULAR DEPENDENCE OF THE SURFACE TENSION

Here we prove inequality (2.8). First of all, we state the corresponding inequality for the Ising model. For simplicity, we fix $d+1=3$, but the results hold for any d. Let $\Lambda = [-L, L]^2 \times [M, M]$ and

$$-H_\Lambda = \sum_{\alpha=1}^{3} J_\alpha \sum_{X \in \Lambda} \sigma_x \sigma_{x+\alpha} \tag{A1}$$

where $\sigma_x = \pm 1$, and the sum over α runs over the natural basis of $\mathbb{Z}^3$. Our results hold for the following two b.c., both denoted by $(0, K)$, $|K| < M$:

$$
\begin{aligned}
\text{(a)}\ \sigma_x &= +1, &&\text{for}\quad x_1 = -L-1,\ x_3 \geqslant 0\\
\sigma_x &= -1, &&\text{for}\quad x_1 = -L-1,\ x_3 < 0\\
\sigma_x &= +1, &&\text{for}\quad x_1 = L+1,\quad x_3 \geqslant \kappa\\
\sigma_x &= -1, &&\text{for}\quad x_1 = L+1,\quad x_3 < \kappa
\end{aligned}
$$

and free b.c. in the x_2 direction; or

$$
\begin{aligned}
\text{(b)}\ \sigma_x &= +1, &&\text{for}\quad x_1 \leqslant 0, &&x_3 \geqslant 0\\
\sigma_x &= -1, &&\text{for}\quad x_1 \leqslant 0, &&x_3 < 0\\
\sigma_x &= +1, &&\text{for}\quad x_1 > 0, &&x_3 \geqslant \kappa\\
\sigma_x &= -1, &&\text{for}\quad x_1 > 0, &&x_3 < \kappa
\end{aligned}
$$

If we let $M \to \infty$ and $J_3 \to \infty$, $J_1 = J_2 = 1$, we obtain the S.O.S. model. The first b.c. above lead to the "Neumann" b.c., while the second b.c. lead to the fixed b.c. (2.4).

We denote by $Z^{\pm}_{\Lambda,\beta}(\kappa)$ the partition function in the Ising model, with b.c. (a) or (b) above, and by $Z^{+}_{\Lambda,\beta}$ the one with $+$ b.c. Following (2.6) and (2.7), we define

$$\tau_\beta(\theta) = -\frac{1}{\beta} \lim_{L\to\infty} \frac{\cos\theta}{(2L+1)^2} \lim_{M\to\infty} \log\left(\frac{Z^{\pm}_{\Lambda,\beta}(\kappa(L))}{Z^{+}_{\Lambda,\beta}}\right) \tag{A2}$$

with

$$\kappa(L) \cong (2L+1)\tan\theta \tag{A3}$$

and

$$\tau_\beta^{\text{step}} = -\frac{1}{\beta} \lim_{L\to\infty} \frac{1}{2L+1} \lim_{M\to\infty} \log\left(\frac{Z^{\pm}_{\Lambda}(1)}{Z^{\pm}_{\Lambda}(0)}\right) \tag{A4}$$

Now we shall prove that

$$\tau_\beta(\theta) - \tau_\beta(0) \geqslant |\sin\theta|\, \tau_\beta^{\text{step}} \tag{A5}$$

for the Ising model (A1) with b.c. (a) or (b). This implies (2.8) for the S.O.S. model which is obtained after taking the limit $J_3 \to \infty$. Using (A2), we write

$$\tau_\beta(\theta) - \tau_\beta(0) = -\frac{1}{\beta} \lim_{L\to\infty} \frac{\cos\theta}{(2L+1)^2} \lim_{M\to\infty} \log\left(\frac{Z^{\pm}_{\Lambda,\beta}(\kappa(L))}{Z^{\pm}_{\Lambda,\beta}(0)}\right) \tag{A6}$$

and

$$\frac{Z^{\pm}_{\Lambda,\beta}(\kappa)}{Z^{\pm}_{\Lambda,\beta}(0)}=\prod_{i=0}^{\kappa-1}\frac{Z^{\pm}(i+1)}{Z^{\pm}(i)} \tag{A7}$$

Inequality (A5) follows from (A6), (A7), the definition (A3) of $\kappa(L)$, the evenness of $\tau_\beta(\theta)$ in θ, and the inequality

$$\lim_{M\to\infty}\frac{Z^{\pm}_{\Lambda,\beta}(1)}{Z^{\pm}_{\Lambda,\beta}(0)}\geqslant\lim_{M\to\infty}\frac{Z^{\pm}_{\Lambda,\beta}(i+1)}{Z^{\pm}_{\Lambda,\beta}(i)} \tag{A8}$$

which we now prove. We choose the b.c. (a) [case (b) is similar]. Observe that if we change, in $Z^{\pm}_{\Lambda,\beta}(i+1)/Z^{\pm}_{\Lambda,\beta}(i)$, the boundary spins $\sigma_x=+1$ for $x_1=-L-1$, $0\leqslant x_3\leqslant i$, into $\sigma_x=-1$ and then take the limit $M\to\infty$ we obtain $\lim_{M\to\infty}Z^{\pm}_{\Lambda,\beta}(1)/Z^{\pm}_{\Lambda,\beta}(0)$. Thus we have to prove that this change of b.c. increases the ratio of partition functions. Since the boundary field induced by a b.c. $\sigma_x=+1$ is equal to $+J_1$, it is enough to prove that

$$\frac{d}{dh}\log[Z^{h}_{\Lambda,\beta}(i+1)/Z^{h}_{\Lambda,\beta}(i)]\leqslant 0 \tag{A9}$$

for $-J_1\leqslant h\leqslant +J_1$, where in Z^h we have the same b.c. as in $Z^{\pm}$, except that we replace the boundary field $+J_1$ induced by $\sigma_x=+1$, for $x_1=-L-1$, $0\leqslant x_3\leqslant i$, by h. The derivative with respect to h, in (A9), equals

$$\sum_{\substack{x:x_1=-L\\0\leqslant x_3\leqslant i}}\{\langle\sigma_x\rangle^{h}_{\Lambda,\beta}(i+1)-\langle\sigma_x\rangle^{h}_{\Lambda,\beta}(i)\}$$

which is negative, termwise.

This last claim follows from F.K.G. inequalities,[40] because the state with b.c. (i) has more + spins on its boundary than the one with b.c. $(i+1)$, and hence dominates the latter state in the F.K.G. sense.

APPENDIX 2: CONTINUUM GAUSSIAN MODEL WITH A CUTOFF

In this appendix we prove our results for the model with Hamiltonian

$$H=\sum_{\langle xy\rangle}(\phi_x-\phi_y)^2$$

and Gibbs state

$$d\mu_\Lambda(\phi)=Z_\Lambda^{-1}\exp[-H_\Lambda(\phi)]\prod_{x\in\Lambda}\chi(|\phi_x|\leqslant l)\,d\phi_x \tag{A10}$$

where $\Lambda \subset \mathbb{Z}^d$. We set $\beta = 1$ (since it can be scaled out by changing l) and impose zero boundary conditions (b.c.) i.e., $\phi_x = 0$, for $x \notin \Lambda$.

We shall also consider, in the course of the proofs, more general single-spin measures, where $\chi(|\phi_x| \leqslant l)$ in (A10) is replaced by some distribution $\rho(\phi_x)$. The most general $\rho(\phi)$ that we shall use is given by

$$\rho(\phi) = \exp[-m^2\phi^2/2 - (\phi/l')^{2n}]\, \chi(|\phi| \leqslant l) \tag{A11}$$

All quantities (free energy or expectation values) defined with respect to this $\rho(\phi)$ depend on four parameters l, m, l', and n. This will be indicated by adding the symbols

$$(l, m, [l', n]) \tag{A11a}$$

In several special cases, it is advantageous to use shorter notation. If $\rho(\phi) = \exp(-m^2\phi^2/2)\, \chi(|\phi| \leqslant l)$ then we write

$$(l, m) \tag{A11b}$$

In particular, if ρ is Gaussian, i.e., $\rho(\phi) = \exp(-m^2\phi^2/2)$, we add the symbol

$$(\infty, m) \tag{A11c}$$

If, in (A11), $m = 0$ and $l = \infty$, then we write

$$[l', n] \tag{A11d}$$

The choice $\rho(\phi) = \chi(|\phi| \leqslant l)$ as in (A10), is indicated by adding the symbol (l). However, within the proper context we will omit this symbol.

Expectation values in a finite box Λ with 0 b.c. are indicated by $\langle(\cdot)\rangle_\Lambda$. We also consider $+l$ and $-l$ b.c., namely, $\phi_x = +l$ (or $-l$), for all $x \notin \Lambda$, as well as periodic b.c., denoted, respectively, by $\langle(\cdot)\rangle_{\Lambda,+l}$, $\langle(\cdot)\rangle_{\Lambda,P}$.

We note that the expectation values

$$\left\langle \prod_x \phi_x \right\rangle_\Lambda, \quad \left\langle \prod_x \phi_x \right\rangle_{\Lambda,+l}, \quad \left\langle \prod_x \phi_x \right\rangle_{\Lambda,-l}$$

converge, as $\Lambda \uparrow \mathbb{Z}^d$, a consequence of Griffiths' inequalities,[31] and define infinite-volume Gibbs states.

Proposition A1 (based on an argument due to Sokal[27]). If

$$\langle \phi_0 \phi_x \rangle = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \phi_0 \phi_x \rangle_\Lambda$$

satisfies

$$\langle \phi_0 \phi_x \rangle \leqslant \exp(-m\,|x|)$$

for some positive m, then, for $\Lambda = [-L, L]^d$, and some finite constant,

$$|\langle \phi_0 \rangle_{\Lambda, \pm l}| \leqslant c L^{d-1} \exp(-mL) \tag{A12}$$

and the infinite-volume Gibbs state is unique. Moreover, (by Ref. 13 or Theorem 3.1), the hypothesis always holds for some $m > 0$.

Proof. (A12) implies that $\langle \phi_0 \rangle_{\Lambda, \pm l} \to 0$, as $\Lambda \uparrow \mathbb{Z}^d$, and this implies the uniqueness of the Gibbs state, by well-known arguments based on the F.K.G. inequalities; see Ref. 49. To prove (A12), we write

$$\langle \phi_0 \rangle_{\Lambda, +l} = \int_0^l \left(\frac{d}{dh} \langle \phi_0 \rangle_{\Lambda, h} \right) dh$$

where h is a boundary field acting on ϕ_x, for $x \in \partial\Lambda \equiv \{x \in \Lambda \mid \exists y \notin \Lambda, |x - y| = 1\}$. More explicitly,

$$\langle \phi_0 \rangle_{\Lambda, +l} = \int_0^l dh \sum_{x \in \partial\Lambda} (\langle \phi_0 \phi_x \rangle_{\Lambda, h} - \langle \phi_0 \rangle_{\Lambda, h} \langle \phi_x \rangle_{\Lambda, h}) \tag{A13}$$

Now we use G.H.S. inequalities,[45] which hold for this model,[50] to bound the truncated two-point functions in (A13) by their values at $h = 0$ (where $\langle \phi_0 \rangle_\Lambda = \langle \phi_x \rangle_\Lambda = 0$). This yields

$$\langle \phi_0 \rangle_{\Lambda, +l} \leqslant l \left(\sum_{x \in \partial\Lambda} \langle \phi_0 \phi_x \rangle_\Lambda \right)$$

from which we obtain (A12), since, by Griffiths' inequalities,[31]

$$\langle \phi_0 \phi_x \rangle_\Lambda \leqslant \langle \phi_0 \phi_x \rangle$$

This completes the proof of Proposition A1. ∎

Now we turn to the following:

Proof of Theorem 3.1. The results for $d = 1$ are fairly standard.[12,51] Using the transfer-matrix formalism, we reduce the problem to the study of an integral operator K with kernel

$$K(\phi_x, \phi_y) = \exp[-(\phi_x - \phi_y)^2] \text{ on } L^2([-l, l])$$

Then $\psi(l)$ is given by the logarithm of the largest eigenvalue of K and $m(l)$ by the logarithm of the ratio between the largest and the second largest eigenvalues of K. $\langle\phi_0^2\rangle$ is given in terms of an expectation with respect to the eigenvector corresponding to the largest eigenvalue. It is well known that these quantities scale with l^2.[51]

Now we prove the theorem for $d=2$ and, at each stage of the proof, indicate the necessary modifications for $d\geqslant 3$. The proof is divided into several parts:

(1) We start with the *upper bound* on $\psi(l)-\psi(\infty)$, and then we prove the *lower bound*. For this lower bound, we shall use the upper bound on $\langle\phi_0^2\rangle(l)$ which is proven in part (3).

(2) We prove the *lower bound* on $m(l)$. This is based on the Lieb–Simon inequality[53] and several integrations by parts (a similar method was used in Ref. 62). Since this proof is rather long, we divide it into two steps, (a) and (b).

We defer the proof of the *upper bound* on $m(l)$ to part (4).

(3) The *upper bound* on $\langle\phi_0^2\rangle(l)$ follows rather easily from the lower bound on $m(l)$ and the infrared bounds, using an idea of Ref. 56. The *lower bound* on $\langle\phi_0^2\rangle(l)$, however, again requires some work (integrations by parts and use of Brascamp–Lieb inequalities[59]).

(4) Finally, the *upper bound* on $m(l)$ follows easily from the lower bound on $\langle\phi_0^2\rangle(l)$. This will complete the proof of Theorem 3.1. Next, we present the details.

(1) The *upper bound* on $\psi(l)-\psi(\infty)$. Consider $\psi(l)-\psi(\infty)$ and write

$$\begin{aligned}0\leqslant\psi(l)-\psi(\infty)=\;&\psi(l,0)-\psi(l,m)\\&+\psi(l,m)-\psi(\infty,m)\\&+\psi(\infty,m)-\psi(\infty,0)\end{aligned}\tag{A14}$$

where the definition of $\psi(l,m)$ follows from (A11b). The mass m will be chosen later (as a function of l).

We have the following bounds:

$$0\leqslant\psi(\infty,m)-\psi(\infty,0)\leqslant cm\tag{A15}$$

and

$$0\geqslant\psi(l,0)-\psi(l,m)\geqslant -cm\tag{A16}$$

for $0<m\leqslant\text{const.}$

Indeed,

$$\psi(\infty, m) - \psi(\infty, 0) = \int_0^m \frac{d}{dm'} \psi(\infty, m')\, dm'$$

$$= \int_0^m m' \langle \phi_0^2 \rangle (\infty, m')\, dm'$$

$$\leqslant cm \tag{A17}$$

since, by explicit computations with Gaussian measures, we know that

$$\begin{aligned} \langle \phi^2 \rangle (\infty, m') &\leqslant c\, |\log m'|, \qquad \text{for} \quad d=2 \\ &\leqslant c, \qquad \text{for} \quad d \geqslant 3 \end{aligned}$$

This proves (A15); (A16) is similar (using monotonicity in l).

To get an *upper bound* on $\psi(l) - \psi(\infty)$ we insert the following inequalities in (A14): $\psi(l, 0) - \psi(l, m) \leqslant 0$, inequality (A15) for $\psi(\infty, m) - \psi(\infty, 0)$, and the following bound on $\psi(l, m) - \psi(\infty, m)$: First, we write it as

$$-\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log \left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_{\Lambda, P} (\infty, m)$$

which follows from the definitions. We can choose periodic b.c. in $\langle (\cdot) \rangle_{\Lambda,P}(m)$, since the thermodynamic free energy does not depend on the b.c. Then we use chessboard estimates,[52] since our interaction is reflection positive, which gives

$$\begin{aligned} &\left(\left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_{\Lambda, P} (m) \right)^{1/|\Lambda|} \\ &\quad \geqslant \langle \chi(|\phi_0| \leqslant l) \rangle_{\Lambda,P}(m) \\ &\quad = 1 - \langle \chi(|\phi_0| \geqslant l) \rangle_{\Lambda,P}(m) \\ &\quad \geqslant 1 - \exp(-cl^2/|\log m|) \end{aligned} \tag{A18}$$

where the last inequality again follows from explicit properties of the $d=2$ massive Gaussian. Combining everything, we have

$$0 \leqslant \psi(l) - \psi(\infty) \leqslant cm + \exp(-cl^2/|\log m|)$$

Choosing

$$m = \exp(-c'l)$$

yields the desired *upper bound.*

For $d \geqslant 3$, we get $\exp(-cl^2)$ in (A18), and we choose $m = \exp(-cl^2)$ to conclude the proof.

Next, we establish the *lower bound* on $\psi(l) - \psi(\infty)$. We note that $\psi(l, m) - \psi(\infty, m) \geqslant 0$ and

$$\psi(l, 0) - \psi(l, m) + \psi(\infty, m) - \psi(\infty, 0)$$
$$= \int_0^m m'[\langle \phi_0^2 \rangle(\infty, m') - \langle \phi_0^2 \rangle(l, m')]\, dm' \tag{A19}$$

For $d = 2$,

$$\langle \phi_0^2 \rangle(\infty, m') \geqslant c\, |\log m'| \tag{A20}$$

while

$$\langle \phi_0^2 \rangle(l, m') \leqslant \langle \phi_0^2 \rangle(l, 0) \leqslant \tilde{c} l \tag{A21}$$

by Griffiths' inequalities and part (3) of Theorem 3.1, which will be proven later. Then, if we choose $m = \exp(-c'l)$ with $c \cdot c' \geqslant \tilde{c} + 1$, and insert (A21) and (A20) in (A19), we obtain our lower bound.

For $d \geqslant 3$, we use another method:

$$\psi(l) - \psi(\infty) = -\lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_\Lambda \quad (m = 0)$$

(with 0 b.c.). Let $\langle (\cdot) \rangle_\lambda$ be the expectation value with a decoupling parameter λ multiplying all terms $\phi_x \cdot \phi_y$, $x \neq y$, in Z_Λ and $\exp(-H_\Lambda)$ on the right-hand side of (A10).

We claim that

$$\frac{d}{d\lambda} \left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_\lambda \leqslant 0 \tag{A22}$$

To prove this, it suffices to show that

$$\frac{\partial}{\partial \lambda} F_n(a, \lambda) \leqslant 0, \qquad \text{for all } a, n \tag{A22$'$}$$

where

$$F_n(a, \lambda) \equiv \log \left\langle \prod_{x \in \Lambda} \exp[-a(\phi_x/l)^{2n}] \right\rangle_\lambda$$

But (A22′) follows by noting that

$$\frac{\partial}{\partial \lambda} F_n(a, \lambda) = \int_0^a \frac{\partial^2}{\partial \lambda\, \partial a'} F_n(a', \lambda)\, da'$$

with

$$\frac{\partial^2}{\partial \lambda\, \partial a} F_n(a, \lambda) = -\sum_x \sum_{jk} \langle (\phi_x/l)^{2n}; \phi_j \phi_k \rangle_{\lambda,a}$$

and applying Griffiths' inequality.

Using (A22), we bound

$$\left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_\Lambda (m=0) = \left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_{\lambda=1}$$

$$\leqslant \left\langle \prod_{x \in \Lambda} \chi(|\phi_x| \leqslant l) \right\rangle_{\lambda=0} \qquad \text{(A22'')}$$

But, for $\lambda = 0$, all the sites are decoupled, and it is easy to compute that, in this case, $\psi_{\lambda=0}(l) - \psi_{\lambda=0}(\infty) \cong e^{-cl^2}$. By (A22″), this is a *lower bound* on $\psi(l) - \psi(\infty)$.

(2) Let us start with the *lower bound* on $m(l)$. For this, we use Lieb–Simon inequalities,[53] which hold for our model.[54] We want to prove that

$$2 \sum_{x \in \partial\Lambda} \langle \phi_0 \phi_x \rangle_\Lambda(l) < 1 \qquad \text{(A23)}$$

for $\Lambda = [-L, L]^d$, with $L \leqslant e^{cl}$, for $d=2$, and $L \leqslant c^{cl^2}$, for $d \geqslant 3$. We start with $d=2$.

We use the formula

$$\chi(|\phi| \leqslant l) = \lim_{n \to \infty} \exp[-(\phi/l)^{2n}]$$

together with some kind of "diagonal argument": First we shall prove [see (a) below] that, if we replace $\chi(|\phi| \leqslant l)$ in (A10) by $\exp[-(\phi/l)^{2n}]$ and let $n = l$, then

$$m([l, n=l]) \geqslant \exp(-cl) \qquad \text{(A24)}$$

Then we prove (A23), for suitable Λ, by using the bound on $m([l, n=l])$ and controlling the difference between the expectation values with the two different single-spin measures $\exp[-(\phi/l)^{2l}]$ and $\chi(|\phi| \leqslant l)$ [see (b) below].

(a) Let us "add" to the Hamiltonian (A10)

$$\frac{m^2}{2}\sum_x \phi_x^2 - \frac{m^2}{2}\sum_x \phi_x^2$$

and let us integrate by parts,[55,62] with respect to the Gaussian measure of mass m (chosen later), the two-point function $\langle\phi_0\phi_x\rangle([l,n])$. This yields

$$\langle\phi_0\phi_x\rangle([l,n]) = C_{0x}^m - 2nl^{-2n}\sum_y C_{0y}^m\langle\phi_x\phi_y^{2n-1}\rangle([l,n]) + m^2\sum_y C_{0y}^m\langle\phi_x\phi_y\rangle([l,n]) \tag{A25}$$

where $C_{0x}^m \equiv \langle\phi_0\phi_x\rangle(\infty, m)$ is the Gaussian two-point function, and we have perturbed the Gaussian, at each site, by $\rho(\phi) = \exp[-(\phi/l)^{2n} + (m^2/2)\,\phi^2]$.

By Griffiths' inequalities,[31]

$$\langle\phi_x\phi_y^{2n-1}\rangle([l,n]) \geqslant \langle\phi_x\phi_y\rangle([l,n])\langle\phi_0^{2n-2}\rangle([l,n])$$

and therefore,

$$\langle\phi_0\phi_x\rangle([l,n]) \leqslant C_{0x}^m \leqslant c\exp(-m\,|x|)$$

provided that

$$2nl^{-2n}\langle\phi_0^{2n-2}\rangle([l,n]) \geqslant m^2$$

Thus, choosing $m^2 = \exp(-cl)$, all we need is a lower bound

$$2nl^{-2n}\langle\phi_0^{2n-2}\rangle([l,n]) \geqslant \exp(-cl) \tag{A26}$$

for $n = l$. We shall derive this lower bound by several integrations by parts. First of all, we apply Griffiths' inequalities,[31] to bound

$$\langle\phi_0^{2n-2}\rangle([l,n]) \geqslant \langle\phi_0^{2n-2}\rangle(\infty, m', [l,n])$$

where we have added a mass term, $\frac{1}{2}(m')^2\sum_x \phi_x^2$, to the Hamiltonian on the right-hand side. We integrate by parts with respect to the Gaussian measure with mass m'. Thus

$$\langle\phi_0^{2n-2}\rangle(\infty, m', [l,n]) = (2n-3)\,C_{00}^{m'}\langle\phi_0^{2n-4}\rangle(\infty, m', [l,n]) - 2nl^{-2n}\sum_y C_{0y}^{m'}\langle\phi_0^{2n-3}\phi_y^{2n-1}\rangle(\infty, m', [l,n]) \tag{A27}$$

Now, our method consists in reintegrating by parts $\langle \phi_0^{2n-4} \rangle$, until we obtain a "Gaussian" term only involving products of covariances, $C^{m'}$, minus a sum of "remainder" terms. Then, using rather simple estimates and a suitable choice of m' as a function of l, we show that the "Gaussian" term is larger than the sum of the remainder terms, and this yields (A26). Let us bound the second term in (A27). By Griffiths' inequalities and Wick's theorem,

$$\begin{aligned}\langle \phi_0^{2n-3} \phi_x^{2n-1} \rangle(\infty, m', [l, n]) &\leqslant \langle \phi_0^{2n-3} \phi_x^{2n-1} \rangle(m') \\ &\leqslant \frac{[4(n-1)]!}{[2(n-1)]!\, 2^{2(n-1)}} [\langle \phi_0^2 \rangle(m')]^{2(n-1)} \\ &\leqslant c^n n^{2n} (\log m')^{2n}\end{aligned}$$

In the second inequality we have bounded factors $\langle \phi_0 \phi_x \rangle(m')$ by $\langle \phi_0^2 \rangle(m')$, and in the third and last inequality we have used Stirling's formula to estimate factorials and the bound

$$\langle \phi_0^2 \rangle(m') \leqslant \tilde{c}\, |\log m'|$$

for $d = 2$. Hence, if we now choose $m' = e^{-\alpha l}$ and set $n = l$, we see that the second term in (A27) is less (in absolute value) than

$$2nl^{-2n}(m')^{-2}\, c^n n^{2n} (\log m')^{2n} = 2lc^l e^{+2\alpha l} (\alpha l)^{2l} \tag{A28}$$

where we have used that

$$\sum_y C_{0y}^{m'} = (m')^{-2}$$

Now we apply (A27) to $\langle \phi_0^{2n-4} \rangle(\infty, m', [l, n])$ and iterate it until we obtain a pure Gaussian term equal to

$$\frac{(2n-2)!}{(n-1)!\, 2^{n-1}} (C_{00}^{m'})^{n-1} \tag{A29}$$

There are $n-1$ terms in the remainder each of which can be bounded by (A28). So the remainder is less than $2l^2 c^l e^{2\alpha l} (\alpha l)^{2l}$, for $n = l$.

The Gaussian term can be bounded from below, again using Stirling's formula and $C_{00}^{m'} = \langle \phi_0^2 \rangle(m') \cong |\log m'|$, by

$$(c')^l\, l^l (\alpha l)^l$$

for $n=l$ and $c' \neq c$ in (A28). Now, choose α small enough, so that

$$\tfrac{1}{2}(c'\alpha)^l > 2l^2 c^l e^{2\alpha l}\alpha^{2l}$$

and the Gaussian term dominates the remainder.

Coming back to the proof of (A26), we note that we have just derived a lower bound

$$\langle \phi_0^{2n-2} \rangle([l, n=l]) \geqslant \tfrac{1}{2}(c')^l\, l^{2l}\alpha^l$$

which clearly proves (A26).

(b) Next, we prove (A23): We write

$$\begin{aligned} \langle \phi_0\phi_x \rangle_\Lambda(l) = \langle \phi_0\phi_x \rangle_\Lambda(l) - \langle \phi_0\phi_x \rangle_\Lambda(l, 0, [\gamma l, n=\gamma l]) \\ + \langle \phi_0\phi_x \rangle_\Lambda(l, 0, [\gamma l, n=\gamma l]) \end{aligned} \tag{A30}$$

where the expectation values on the right-hand side of (A30) have been defined in (A11a).

By Griffiths' inequalities,

$$\langle \phi_0\phi_x \rangle_\Lambda(l, 0, [\gamma l, n=\gamma l]) \leqslant \langle \phi_0\phi_x \rangle_\Lambda([\gamma l, n=\gamma l])$$

where, on the right-hand side, we have removed the cutoffs $|\phi_x| \leqslant l$. By part (a), (A24),

$$m([\gamma l, n=\gamma l]) \geqslant \exp(-c\gamma l)$$

and therefore

$$\begin{gathered} \sum_{x\in\partial\Lambda} \langle \phi_0\phi_x \rangle_\Lambda(l, 0, [\gamma l, n=\gamma l]) \ll 1 \\ \text{for } \Lambda=[-L, L]^2, \qquad \text{with } L \cong \exp(c'l)\ (c' < c\gamma) \end{gathered} \tag{A31}$$

To bound

$$\sum_{x\in\partial\Lambda} \langle \phi_0\phi_x \rangle_\Lambda(l) - \langle \phi_0\phi_x \rangle_\Lambda(l, 0, [\gamma l, n=\gamma l]) \tag{A32}$$

we simply replace $\exp[-(\phi/\gamma l)^{2n}]$ in (A30) by $\exp[-\lambda(\phi/\gamma l)^{2n}]$ and differentiate with respect to λ. This yields

$$\begin{aligned} (\text{A32}) &= \sum_{x\in\partial\Lambda} \int_0^1 d\lambda \left(\frac{d}{d\lambda} \langle \phi_0\phi_x \rangle_\Lambda^\lambda \right) \\ &= (\gamma l)^{-2n} \sum_{\substack{x\in\partial\Lambda \\ y\in\Lambda}} \int_0^1 d\lambda [\langle \phi_0\phi_x\phi_y^{2n} \rangle_\Lambda^\lambda - \langle \phi_0\phi_x \rangle_\Lambda^\lambda \cdot \langle \phi_y^{2n} \rangle_\Lambda^\lambda] \end{aligned}$$

and the notation $\langle (\cdot) \rangle_\Lambda^\lambda$ is self-explanatory.

Using $n=\gamma l$ and $|\phi_x|\leqslant l$, because of the cutoff in (A32), we get that

$$\begin{aligned}(\text{A32}) &\leqslant \tilde{c}(\gamma l)^{-2\gamma l}\,|\Lambda|\,|\partial\Lambda|\,l^{2\gamma l+2}\\ &\leqslant c(\gamma l)^{-2\gamma l}\exp(3c'l)\,l^{2\gamma l+2}\end{aligned}\tag{A33}$$

using (A31).

If we choose γ large enough we can make (A33) as small as we wish, and (A23) is proven.

For $d\geqslant 3$, we modify our argument as follows. Let $n=l^2$ in part (a), and keep in mind the fact that $C_{00}^{m'}$ is bounded uniformly in m' in $d\geqslant 3$. Choosing $m'=\exp(-cl^2)$, we obtain $\langle\phi_0\phi_x\rangle([l,n=l^2])\leqslant C_{0x}^{m'}$. Part (b) of the argument is as above, provided that we choose $n=\gamma l^2$. Of course, this lower bound on $m(l)$ also follows from Ref. 13.

Now, in order to complete the proof of part (2) of the Theorem, we still have to prove the *upper bound* on $m(l)$. However, we shall first prove part (3) of the Theorem and then use it to prove this upper bound.

(3) We start by proving the *upper bound* on $\langle\phi_0^2\rangle(l)$. Such a bound follows rather easily from the lower bound on $m(l)$ that we just proved. Indeed, the larger the mass, the smaller the variance of ϕ_0. We shall adapt an argument of Ref. 56, Theorem 6.2, to our situation. Using the infrared bounds,[57] we know that

$$\begin{aligned}\langle(\phi_0-\phi_x)^2\rangle(l) &\leqslant \langle(\phi_0-\phi_x)^2\rangle \qquad (m=0)\\ &\cong c\ln|x|, \qquad \text{for}\quad d=2\end{aligned}$$

Furthermore, reflection positivity implies a spectral representation for the two-point function,[58] from which it easily follows that

$$\langle\phi_0\phi_x\rangle(l)\leqslant\langle\phi_0^2\rangle(l)\exp[-m(l)\,|x|]$$

Thus, for all x, we have that

$$2\langle\phi_0^2\rangle(l)\{1-\exp[-m(l)\,|x|]\}\leqslant c\ln|x| \tag{A34}$$

Now, set $|x|=\exp(cl)$ in (A34) and recall that $m(l)\geqslant\exp(-cl)$ [see part (2)], to obtain $\langle\phi_0^2\rangle(l)\leqslant c'l$. The $d=1$ bound is trivial, since $|\phi_0|\leqslant l$, and the $d\geqslant 3$ case will follow from our proof of the *lower bound* on $\langle\phi_0^2\rangle(l)$ which we now present.

By Griffiths' inequalities,[31]

$$\langle\phi_0^2\rangle(l)\geqslant\langle\phi_0^2\rangle(l,m) \tag{A35}$$

where we have inserted a mass term of strength m^2 on the right-hand side.

Integrating by parts with respect to the Gaussian measure with mass m gives

$$\begin{aligned}\langle\phi_0^2\rangle(l, m) &= C_{00}^m - \sum_x C_{0x}^m \langle\phi_0[\delta(\phi_x = l) - \delta(\phi_x = -l)]\rangle(l, m)\\ &\geqslant C_{00}^m - 2m^{-2} l\langle\delta(\phi_0 = l)\rangle(l, m) \end{aligned}\tag{A36}$$

Now we estimate $\langle\delta(\phi_0 = l)\rangle(l, m)$. Since $\chi(|\phi| \leqslant l)$ is a log-concave function of ϕ, we know, from Brascamp–Lieb inequalities,[59] that the marginal distribution of ϕ_0 in $\langle(\cdot)\rangle(l, m)$ is of the form

$$P(\phi_0) = \exp(-\alpha_m \phi_0^2)\, G(\phi_0)$$

with $\int_{-l}^{+l} P(\phi_0)\, d\phi_0 = 1$. Moreover, $\exp(-\alpha_m \phi_0^2)$ is (up to a constant factor) the marginal distribution in the Gaussian measure of mass m. Thus $\alpha_m \cong 1/(c\,|\log m|)$, for $d = 2$. Finally, $G(\phi_0)$ is even, log-concave, and thus monotone decreasing on $\mathbb{R}_+$. Hence, we obtain

$$\langle\delta(\phi_0 = l)\rangle(l, m) = \exp(-\alpha_m l^2)\, G(l) \tag{A37}$$

and we may bound $G(l)$ by $G(1)$, for $l \geqslant 1$, by monotonicity. Moreover,

$$G(1) = \tfrac{1}{2}[G(1) + G(-1)] \leqslant \int_{-1}^{+1} G(\phi)\, d\phi$$

by evenness and monotonicity. Hence

$$\begin{aligned} G(l) &\leqslant \exp \alpha_m \int_{-l}^{+l} \exp(-\alpha_m \phi^2)\, G(\phi)\, d\phi \\ &= \exp \alpha_m \int_{-l}^{+l} P(\phi)\, d\phi = \exp \alpha_m \end{aligned}$$

This bound on $G(l)$ and (A37) together give

$$\langle\delta(\phi_0 = l)\rangle(l, m) \leqslant \exp[-(l^2 - 1)/(c\,|\log m|)]$$

Inserting this into (A36) and choosing $m = \exp(-c''l)$, with c'' small enough, yields

$$\langle\phi_0^2\rangle(l, m) \geqslant c'l, \qquad \text{for} \quad d = 2$$

which, by (A35), concludes the proof in the two-dimensional case.

For $d \geqslant 3$, we use the same argument, but we can integrate directly with respect to $m = 0$ in (A36). This finishes the proof of part (3) of Theorem 3.1.

(4) Now we complete the proof of part (2) by proving the *upper bound* on $m(l)$. To this end, insert the lower bound, $\langle\phi_0^2\rangle \geqslant c'l$ (that we have just proven) into (A34), and choose $|x| = \exp(c'l/c)$. This yields the bound for $d=2$. For $d \geqslant 3$, the infrared bounds[57] give

$$\langle(\phi_0-\phi_x)^2\rangle(l) \leqslant \langle\phi_0^2\rangle(m=0)(1-c/|x|^{d-2})$$

Thus, instead of (A34), we now have

$$\langle\phi_0^2\rangle(l)(1-\exp[-m(l)\,|x|]) \leqslant \langle\phi_0^2\rangle(m=0)[1-c/|x|^{d-2}]$$

Using that

$$\langle\phi_0^2\rangle(l) \geqslant \langle\phi_0^2\rangle(m=0) - \exp(-c'l^2)$$

[part (3)], and choosing $|x| = \exp(\tilde{c}l^2)$, for $\tilde{c}$ large enough, finishes the proof. ∎

APPENDIX 3: DISCRETE GAUSSIAN AND S.O.S. MODELS

In the first part of this appendix, we prove Theorem 3.2, which gives estimates on the free energy of the discrete Gaussian and S.O.S. models. This is essentially an extension of some of the results of Appendix 2 to the discrete models.

In the second part, we prove Theorem 4.1. Namely, we show that, because of entropic repulsion, the average height of a surface lying above a wall and tied to it at the boundary of a box Λ diverges logarithmically with the size of Λ.

Part 1: Proof of Theorem 3.2. We use indices D or C to distinguish between the expectation values $\langle(\cdot)\rangle_D$ for the discrete ($\phi \in \mathbb{Z}$) model and the expectation value $\langle(\cdot)\rangle_C$ for the continuum ($\phi \in \mathbb{R}$) model.

For the upper bound on $\psi_\beta^0(l) - \psi_\beta^0(\infty)$ we follow, step by step, the proof of Theorem 3.1, part (1) (Appendix 2). For $d=1$, see Ref. 12. In order to bound $\langle\phi_0^2\rangle_D(\infty, m')$ [see (A17)], we use the correlation inequalities of Ref. 34, Corollary 3.2, which imply that this is less than $\langle\phi_0^2\rangle_C(\infty, m')$, for the continuum ($\phi \in \mathbb{R}$) Gaussian model. Thus, we obtain (A15), (A16), and we only need an upper bound on

$$\langle\chi(|\phi_0| \geqslant l)\rangle_D(\infty, m) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle\chi(|\phi_0| \geqslant l)\rangle_{\Lambda,P,D}(\infty, m)$$

and then choose m as a function of l in a suitable way. For $d \geqslant 2$ and low temperatures, we know, by a standard Peierls argument,[1,2] that

$$\lim_{m \downarrow 0} \langle\chi(|\phi_0| \geqslant l)\rangle_D(\infty, m) \leqslant \exp(-c\beta l^2)$$

and we can take $m(l) = \exp(-c\beta l^2)$. For $d = 2$ and arbitrary β, however, we use

$$\begin{aligned}\langle\chi(|\phi_0| \geqslant l)\rangle_D(\infty, m) &= \langle\exp(c\phi_0 - c\phi_0)\,\chi(|\phi_0| \geqslant l)\rangle_D(\infty, m)\\ &\leqslant \exp(-cl)\langle\exp c\phi_0\rangle_D(\infty, m)\\ &\leqslant \exp(-cl)\langle\exp c\phi_0\rangle_C(\infty, m)\end{aligned}$$

in the continuum Gaussian model (by the inequalities of Ref. 34). Hence

$$\langle\chi(|\phi_0| \geqslant l)\rangle_D(\infty, m) \leqslant \exp(-cl)\exp(c'\,|\log m|)$$

We choose $m(l) = \exp(-c''l)$ with c'' small enough, to conclude the proof of part (2), when $d = 2$.

For $d \geqslant 3$, we again set $m = \exp(-cl^2)$ and we note that, writing $\lim_{m\downarrow 0}\langle(\cdot)\rangle_D(\infty, m) \equiv \langle(\cdot)\rangle_D$, we have

$$\langle\chi(|\phi_0| \geqslant l)\rangle_D \leqslant \exp(-cl^2)\langle\exp(c\phi_0^2)\rangle_D$$

We write $\log\langle\exp(c\phi_0^2)\rangle_D = c\langle\phi_0^2\rangle_D(\lambda c)$, for some $0 \leqslant \lambda \leqslant 1$, by the mean value theorem. Here $\langle(\cdot)\rangle_D(\lambda c)$ is the same as $\langle(\cdot)\rangle_D$, except for a factor $\exp(\lambda c\phi_0^2)$ (in the numerator and in the denominator). By Griffiths' inequalities,[31] and the inequalities in Ref. 34

$$\langle\phi_x^2\rangle_D(\lambda c) \leqslant \langle\phi_x^2\rangle_D(c) \leqslant \langle\phi_x^2\rangle_C(c)$$

Now

$$\langle\phi_x^2\rangle_C(c) \equiv \frac{\langle\phi_x^2 e^{c\phi_0^2}\rangle_C(\infty, m=0)}{\langle e^{c\phi_0^2}\rangle_C(\infty, m=0)}$$

is bounded, for $d \geqslant 3$ and c small enough [namely, $c < 1/2\langle\phi_0^2\rangle_C(\infty, m=0)$]. Thus we have shown that

$$\langle\chi(|\phi_0| \geqslant l)\rangle_D \leqslant c'\exp(-cl^2)$$

which proves the upper bound in part (3). For the lower bounds, we can use the same method as the one used in the proof of Theorem 3.1 (for $d \geqslant 3$).

For the S.O.S. model, we can prove (3.18) by the same method, since we know, by a Peierls' argument[1,2] that, for β large,

$$\lim_{m\downarrow 0}\langle\chi(|\phi_0| \geqslant l)\rangle_D(\infty, m) \leqslant \exp(-c\beta l) \qquad \blacksquare$$

Part 2: Proof of Theorem 4.1. We start, in part (A) below, with the easy half, namely, the *upper bound.* In part (B), we first explain our general strategy in the proof of the lower bound. Then we introduce the necessary definitions and some of the intuitive ideas behind the proof. After this, we state our main estimate (Proposition A2) and, using it, prove the theorem (which is rather easy, given Proposition A2 and some ideas going back to Ref. 63). The remainder of this appendix is then devoted to the proof of Proposition A2.

We shall denote by $\langle(\cdot)\rangle_\Lambda(h, h')$ the expectation value in Λ, with b.c. $\phi_x = h$, $x \notin \Lambda$, and with the constraint $\phi_x \geqslant h'$, $x \in \Lambda$ [with Hamiltonian (4.1) and $a = 0$]. However, we still denote by $\langle(\cdot)\rangle_\Lambda$ the expectation considered in Section 4, i.e., with $h = h' = 0$.

(A) The Upper Bound. By F.K.G. inequalities[40] (which hold for this model; see Ref. 60),

$$\langle\phi_x\rangle_\Lambda \leqslant \langle\phi_x\rangle_\Lambda(h, 0) = h + \langle\phi'_x\rangle_\Lambda(0, -h) \tag{A38}$$

where we changed variables: $\phi_x = h + \phi'_x$. The ϕ'_x variables have b.c. 0 and a constraint $\phi'_x \geqslant -h$. Note, for later use, that, since $\chi(\phi' \geqslant -h)$ is increasing in ϕ', F.K.G. inequalities imply that $\langle\phi'_x\rangle(0, -h) \geqslant 0$, and by (A38), that

$$\langle\phi_x\rangle_\Lambda(h, 0) \geqslant h \tag{A39}$$

Let A be the event that $\phi'_x \geqslant -h$, for all $x \in \Lambda$, and consider the expectation value $\langle(\cdot)\rangle_\Lambda(0, -\infty)$ with no constraint on ϕ'_x. Then $\langle\phi'_x\rangle_\Lambda(0, -\infty) = 0$, by symmetry, and, by conditioning,

$$\begin{aligned} 0 = \langle\phi'_x\rangle_\Lambda(0, -\infty) = {} & \langle\phi'_x\rangle_\Lambda(0, -h)\, \bar{P}_\Lambda(A) \\ & + \langle\phi'_x\rangle_\Lambda(0, A^c)\, \bar{P}_\Lambda(A^c) \end{aligned}$$

or

$$\langle\phi'_x\rangle(0, -h) = -\langle\phi'_x\rangle(0, A^c)\, \bar{P}_\Lambda(A^c)/[1 - \bar{P}_\Lambda(A^c)] \tag{A40}$$

Indeed, conditioning on A amounts to the same as imposing the constraint $\phi'_x \geqslant -h$, $\forall x \in \Lambda$. Clearly $\bar{P}_\Lambda$ denotes the probability determined by the state $\langle(\cdot)\rangle_\Lambda(0, -\infty)$. For this unconstrained system, we know that at low temperatures[1,2]

$$\bar{P}_\Lambda(\phi'_x \leqslant -n) \leqslant \exp(-c\beta n^\alpha) \tag{A41}$$

Now, conditioning on A^c means that there exists an $x \in \Lambda$ with $\phi_x \leqslant -h$. Clearly, if we condition on the smallest ϕ_y, $y \in \Lambda$, to be equal to $-n \leqslant -h$, then

$$\langle\phi'_x\rangle(0, \min_{y \in \Lambda} \phi'_y = -n) \geqslant -n$$

and

$$\bar{P}_\Lambda(\min_{y\in\Lambda} \phi'_y = -n) \leqslant |\Lambda| \max_{y\in\Lambda} \bar{P}_\Lambda(\phi'_y = -n)$$
$$\leqslant |\Lambda| \exp(-c\beta n^\alpha)$$

by (A41).

Now the upper bound in Theorem 4.1 follows if we insert the last two estimates in (A40), (A40) in (A38), and choose $h = (c'/\beta \log |\Lambda|)^{1/\alpha}$, with c' large enough, so that

$$|\langle\phi'_x\rangle(0, A^c)|\, \bar{P}_\Lambda(A^c) \leqslant |\Lambda| \sum_{n=h}^{\infty} n \exp(-c\beta n^\alpha)$$
$$\cong |\Lambda| \exp(-c\beta h^\alpha) \to 0, \qquad \text{as } |\Lambda| \to \infty \quad \text{(A42)}$$

(B) Proof of the Lower Bound. Our general strategy consists in finding a sequence of events A_K such that $P_\Lambda(\bigcup_K A_K) \cong 1$, where P_Λ is given by $\langle(\cdot)\rangle_\Lambda(0, 0)$, and, for each A_K, choosing a region

$$\Lambda_K, \text{ with } |\Lambda_K|/|\Lambda| \cong 1$$

such that

$$\langle\phi_x\rangle(A_K) \geqslant (c/\beta) \log |\Lambda|, \qquad \text{for} \quad x \in \Lambda_K$$

Using $\phi_x \geqslant 0$, for $x \notin \Lambda_K$, or for the events not in $\bigcup_K A_K$, gives the lower bound in the theorem.

To simplify our notations, we shall set $d = 2$ and $\alpha = 1$ in (4.1). We shall describe each configuration $(\phi_x)_{x\in\Lambda}$ in terms of *walls* (and ceilings): Two bonds $\langle xy\rangle$, $\langle x'y'\rangle$ are *adjacent* if the distance between them is less than or equal to $\sqrt{2}$. A set of bonds is *connected* iff any pair of bonds in that set can be linked by a sequence of adjacent bonds. For each configuration $(\phi_x)_{x\in\Lambda}$, we decompose the set of bonds into maximal connected sets where $\phi_x \neq \phi_y$. A *wall* is a pair $w = (\mathbf{w}, \phi_\mathbf{w})$, where $\mathbf{w}$ is such a set of bonds, and $\phi_\mathbf{w}$ is the restriction of the configuration to $\mathbf{w}$. A *ceiling* is a connected component of the complement of $\mathbf{w}$. For a given wall, there is one *exterior ceiling* and one or more finite, *interior ceilings*.

The energy of a wall is

$$E(w) = \sum_{\langle xy\rangle \in \mathbf{w}} |\phi_x - \phi_y|$$

and we can write (4.1) as

$$H_\Lambda = \sum_w E(w)$$

Clearly,

$$E(w) \geq |\mathbf{w}| = \text{number of bonds in } \mathbf{w} \tag{A43}$$

The walls are similar to the contours in the Ising model and are the basis of a low-temperature expansion for the S.O.S. model without constraints.[2] Here we shall use a different notion of contours (related to the one used in Ref. 41) which are defined on a "large scale," i.e., a notion which depends on Λ itself. The intuitive motivation for the introduction of these new contours is as follows. Usually the contours of a configuration correspond to those regions, where the energy of the configuration is large, and therefore they have a small probability, for β large. Here we must deal with an additional, totally different effect: If the surface is not very high (with respect to $\phi = 0$) it looses a lot of low-energy excitations ("spikes") in comparison to a higher-lying surface. We want to make use of these low-energy excitations, in order to prove that it is improbable for a surface to be close to the wall, and therefore we want to include in our notion of contours the regions where the height of the surface is anomalously small. However, since the weight of a spike is small [$=\exp(-4\beta p)$, for a spike of height p] at low temperatures, we shall only be able to take advantage of this damping factor if the surface is low over a large region: If a surface has height $p-1$ over a region $\tilde{\Lambda}$, then the loss of free energy is roughly (see Section 4.2) $|\tilde{\Lambda}| \exp(-4\beta p)$, and this term is large only for $|\tilde{\Lambda}| \gg \exp(4\beta p)$.

Let us now define the "large-scale" contours. First of all, we fix a number n (large enough) and prove the lower bound only for regions $\Lambda = \Lambda_q = [-L_q, L_q]^2$, with $|\Lambda_q| = \exp(\beta q)$, where $q = n \cdot m$, and $m \in \mathbb{N}$ is large enough.

The fact that, by F.K.G. inequalities,[40] $\langle \phi_x \rangle_\Lambda$ is increasing in Λ [if $\Lambda \subset \Lambda'$, setting $\phi_x = 0$ on $\Lambda' \backslash \Lambda$ amounts to multiplying by the function $\prod_{x \in \Lambda' \backslash \Lambda} \chi(\phi_x \leq 0)$ which is decreasing] allows us to restrict our attention to the regions Λ_q. Let $B_0 \equiv [-b, b[^2$, with $|B_0| = \beta \exp(4\beta m)$ (i.e., the size of B_0 depends on Λ, namely, $|B_0| = \beta\, |\Lambda|^{4/n}$). We cover Λ with translates, B_i, of B_0, $B_i \equiv B_0 + ib$, $i \in \mathbb{Z}^2$ (these boxes overlap for $|i-j| \leq \sqrt{2}$).

Given a configuration $(\phi_x)_{x \in \Lambda}$, we call a box B_i *regular* iff (i) for all walls, w, such that $\mathbf{w} \cap B_i \neq \varnothing$,

$$E(w) \leq 4m = (4/\beta n) \log |\Lambda| \tag{A44}$$

(ii) $\phi_x \geq m$, for all $x \in B_i$ that do not belong to an interior ceiling of a wall w, with $\mathbf{w} \cap B_i \neq \varnothing$.

Notice that ϕ_x is constant on the set of sites, x, specified in (ii).

A *contour* is a *maximal, connected set of irregular boxes* (two boxes B_i, B_j being "adjacent" if $|i-j| \leq \sqrt{2}$ i.e., if they overlap). When no confusion arises, we shall use the same letter Γ to denote a contour and the corresponding subset of the lattice $\bigcup_{B \in \Gamma} B$.

Now we observe that our b.c. $\phi_x = 0$ and our previous definitions imply that all the boxes outside Λ are irregular. Let Γ be the contour inside Λ which is connected to these irregular boxes; $\|\Gamma\|$ denotes the number of boxes in Γ, and P_Λ the probability distribution for surfaces over $\Lambda = [-L, L]^2$ with b.c. $\phi_x = 0$, $x \notin \Lambda$. Then our main estimate is the following:

Proposition A2:

$$P_\Lambda(\|\Gamma\| \geqslant L^{3/2}) \to 0, \qquad \text{as} \quad L \to \infty \tag{A45}$$

It should be noted that the exponent $\frac{3}{2}$ on the left-hand side of (A45) could be replaced by any $\gamma \in (1, 2)$.

Proof of Theorem 4.1, Given Proposition A2. The events A_K mentioned at the beginning of (B) (proof of lower bound) will be defined to be the different contours Γ with $\|\Gamma\| \leqslant L^{3/2}$ and $\Lambda_K = \Lambda \setminus \bar{\Gamma}$, where $\bar{\Gamma} = \Gamma \cup \{\text{regular boxes intersecting } \Gamma\}$. Clearly, $|\Lambda_K| \cong 4L^2 \cong |\Lambda|$, as $L \to \infty$, because $|\bar{\Gamma}|$, the number of sites in $\bar{\Gamma}$ is of the order of $\|\bar{\Gamma}\|\ |B| \cong \|\Gamma\|\ \beta \exp(4\beta m) \cong \beta L^{3/2} L^{4/n} \ll L^2$, for n large enough. We have still to bound $\langle \phi_x \rangle(\Gamma)$, conditioned on Γ, for $x \in \Lambda \setminus \bar{\Gamma}$. Since Γ is connected to the exterior of Λ, if $x \notin \bar{\Gamma}$ then x belongs to one of the interior components surrounded by Γ, which we denote by V. The boundary of these interior parts, ∂V, is defined as the (connected) set of regular boxes in V that intersect Γ (i.e., they belong to $\bar{\Gamma} \setminus \Gamma$). From the definition of a regular box it is easy to deduce that we can find a closed path of nearest neighbors in $\mathbb{Z}^2$:

$$\pi = \{(x_i)_{i=1}^n \mid |x_{i+1} - x_i| = 1,\ x_n = x_1\}$$

such that (i) $\pi \subset \partial V$; (ii) $V \subset \operatorname{Int} \pi$, where $\operatorname{Int} \pi$ denotes the region interior to π [here $d = 2$; in general, π will be a $(d-1)$-dimensional surface]; and (iii)

$$\phi_{x_i} \geqslant m, \qquad \forall x_i \in \pi \tag{A46}$$

Indeed, in a regular box, the walls are small compared to the size of the box; ($|\mathbf{w}| \leqslant E(w) \leqslant \log |\Lambda| \ll \beta\ |\Lambda|^{4/n} = |B|$). Therefore, any side of B can be joined to any other side by a sequence of points outside the walls. This property also holds for a connected set of regular boxes, such as ∂V.

Notice that, since the interaction is nearest neighbor, conditioning on the values of ϕ_x along a closed path π decouples $\operatorname{Int} \pi$ from the rest. With this in mind, we write

$$\langle \phi_x \rangle_\Gamma = \sum_\pi P_\Gamma(\pi) \cdot \langle \phi_x \rangle(\pi) \tag{A47}$$

where we condition on the outermost path, π, satisfying the three properties (A46). This conditioning is equivalent to requiring b.c. $\phi_x \geqslant m$ on Int π, because choosing the outermost path does not impose any condition on the spins in Int π.

By (A39), this implies $\langle \phi_x \rangle(\pi) \geqslant m \cong (c/\beta) \log |\Lambda|$, and, by (A47), $\langle \phi_x \rangle_\Gamma \geqslant (c/\beta) \log |\Lambda|$, if $x \in V$.

This completes the proof of the desired lower bound. ∎

Proof of Proposition A2. First of all, we make two changes in the partition function appearing in the denominator of $P_\Lambda(\cdot)$. We start by replacing the b.c. $\phi_x = 0$ by $\phi_x = h \equiv (c/\beta) \log |\Lambda|$, with c large enough. This produces an error of at most $\exp(\beta h |\partial\Lambda|) \cong \exp[c(\log |\Lambda|) \cdot |\partial\Lambda|]$, due to bonds across $\partial\Lambda$. Once we have chosen the b.c. $\phi_x = h$ we change variables,

$$\phi'_x = \phi_x - h$$

and we remove the constraint $\phi_x \geqslant 0$, which, for the variables ϕ'_x, is $\phi'_x \geqslant -h$.

We use the notation $Z_\Lambda(h, h')$ for the partition function with b.c. $\phi_x = h$, $x \in \partial\Lambda$, and constraint $\phi_y \geqslant h'$, for all $y \in \Lambda$. Furthermore, $Z_\Lambda(h, h' \mid \Gamma)$ denotes the partition function with the same b.c. and constraint, conditioned on the presence of the contour Γ. The above argument then shows that

$$\begin{aligned} P_\Lambda(\Gamma) &= \frac{Z_\Lambda(0, 0 \mid \Gamma)}{Z_\Lambda(0, 0)} \\ &\leqslant c' \exp(c\, |\partial\Lambda| \log |\Lambda|) \cdot \frac{Z_\Lambda(0, 0 \mid \Gamma)}{Z_\Lambda(0, -\infty)} \cdot \frac{Z_\Lambda(0, -\infty)}{Z_\Lambda(0, -h)} \end{aligned}$$

We now note that

$$\begin{aligned} \frac{Z_\Lambda(0, -h)}{Z_\Lambda(0, -\infty)} &= 1 - \bar{P}_\Lambda(\min_{y \in \Lambda} \phi_y < -h) \\ &\geqslant \text{const} > 0 \end{aligned}$$

provided the constant c in our choice of h is large enough. This follows directly from (A41), as in the proof of (A42). Hence

$$P_\Lambda(\Gamma) \leqslant \tilde{c} \exp(c\, |\partial\Lambda| \log |\Lambda|) \cdot \frac{Z_\Lambda(0, 0 \mid \Gamma)}{Z_\Lambda(0, -\infty)} \tag{A.48}$$

Now we shall prove that the last ratio of partition functions is less than $\exp(-c\beta\, \|\Gamma\|)$. This will imply (A45), since $|\partial\Lambda| \log |\Lambda| \ll L^{3/2}$, for L

large (here $d=2$; the exponent 3/2 could be replaced by any γ, with $1<\gamma<2$). There are two kinds of irregular boxes in Γ, depending on which of the two conditions, (i) and (ii), defining a regular box is violated: Either there exists a w with $\mathbf{w}\cap B\neq\varnothing$ and $E(w)\geqslant 4m+1$; or (A44) holds, but $\phi_x=p\leqslant m-1$, for the sites $x\in B$ which are in the exterior of all the walls in B. We denote by Γ_1, resp. Γ_2, the subset of Γ composed of the first, resp. the second type of irregular boxes.

We note that

$$Z_\Lambda(0,0\mid\Gamma)\leqslant\sum_{\{w_1,\ldots,w_n\}}\tilde{Z}_\Lambda(w_1,\ldots,w_n\mid\Gamma_2)$$

where the sum extends over all sets of walls $\{w_i\}_{i=1}^n$, with $E(w_i)\geqslant 4m+1$ and $\mathbf{w}_i\cap B\neq\varnothing$ for some $B\in\Gamma_1$, for all i. The inequality comes from the fact that we relax in $\tilde{Z}_\Lambda(w_1,\ldots,w_n\mid\Gamma_2)$ the constraint $\phi_x\geqslant 0$ and sum over all configurations (still with 0 b.c.), except those containing some downwards spikes of height m localized in a box $B\in\Gamma_2$. This means that we exclude the configurations, where, for some $x\in B$ ($B\in\Gamma_2$), $\phi_x=p-m$, and $\phi_y=p$, for all y with $|y-x|=1$. In other words, all possible walls are allowed except such downwards spikes, which do not appear in $Z_\Lambda(0,0\mid\Gamma)$ either, by definition of Γ_2. Now we can write a convergent low-temperature expansion (as in Ref. 2) for $\log\tilde{Z}_\Lambda(w_1,\ldots,w_n\mid\Gamma_2)$ and for $\log Z_\Lambda(0,-\infty)$: For example,

$$\log Z_\Lambda(0,-\infty)=\sum_{W\subset\Lambda}\phi^T(W)\exp[-\beta E(W)]$$

where the sum runs over all multiplicity functions, W, defined on the set of walls in Λ, the $\phi^T(W)$ are the usual truncated functions,[1,61] and $E(W)=\sum_w E(w)\,W(w)$ [sum over all walls: $W(w)\equiv$ multiplicity of w in W].

As a result of those expansions we obtain

$$\begin{aligned}\log\tilde{Z}_\Lambda(w_1,\ldots,w_n\mid\Gamma_2)&-\log Z_\Lambda(0,-\infty)\\&\leqslant-\beta\sum_{i=1}^n E(w_i)-c\exp(-4\beta m)\,|B|\,\|\Gamma_2\|\\&\quad+c'\exp(-4\beta)\sum_{i=1}^n|\mathbf{w}_i|\end{aligned}\tag{A49}$$

where the first term is simply the total energy of the walls w_i coming from $\tilde{Z}_\Lambda(w_1,\ldots,w_n\mid\Gamma_2)$; the second term comes from the absence of the spikes of height m in boxes of Γ_2, when evaluating $\tilde{Z}_\Lambda(w_1,\ldots,w_n\mid\Gamma_2)$—which are summed over when one calculates $Z_\Lambda(0,-\infty)$. The volume of the region where those spikes are suppressed is $|B|\cdot\|\Gamma_2\|$. The third term in (A49) is a correction coming from the fact that in evaluating $Z_\Lambda(0, -\infty)$ we also sum over the walls w such that $\mathbf{w}$ intersects $\mathbf{w}_i$, for some $i=1,..., n$, and moreover from a boundary term in the expansion of $\log \tilde{Z}_\Lambda(w_1,..., w_n \mid \Gamma_2)$. This correction is proportional to the largest weight of a wall $[\exp(-4\beta)]$ times the total length of the walls $w_1,..., w_n$. Concerning the other boundary terms in the expansion, those along $\partial\Lambda$, they cancel since we have the same b.c. in both partition functions. Using (A49), and the size of $|B| = \beta\exp(4\beta m)$ we get

$$\frac{Z_\Lambda(0, 0 \mid \Gamma)}{Z_\Lambda(0, -\infty)} \leqslant \sum_{\{w_1,...,w_n\}} \frac{\tilde{Z}_\Lambda(w_1,..., w_n \mid \Gamma_2)}{Z_\Lambda(0, -\infty)}$$
$$\leqslant \sum_{\{w_1,...,w_n\}} \exp\left[-\beta \sum_{i=1}^{n} \tilde{E}(w_i)\right]$$
$$\times \exp(-c\beta \, \|\Gamma_2\|) \tag{A50}$$

where

$$\tilde{E}(w) \equiv E(w) - c'/\beta \exp(-4\beta) \sum_{i=1}^{n} |\mathbf{w}_i| \tag{A51}$$

with c' as in (A49). For β large, $\tilde{E}(w)$ is a small perturbation of $E(w)$ [see (A43)].

Now we shall sum the factor $\exp[-\beta \sum_{i=1}^{n} \tilde{E}(w_i)]$ in (A50) over $\{w_1,..., w_n\}$. Since each $\mathbf{w}_i$ intersects some $B \in \Gamma$, we have

$$\sum_{\{w_1,...,w_n\}} \exp\left[-\beta \sum_{i=1}^{n} \tilde{E}(w_i)\right]$$
$$\leqslant \prod_{B \in \Gamma_1} \left[\sum_{n=1}^{\infty} \frac{1}{n!} \Bigg(\sum_{\substack{w:\mathbf{w} \cap B \neq \varnothing \\ E(w) \geqslant 4m+1}} \exp[-\beta \tilde{E}(w)] \Bigg)^n\right] \tag{A52}$$

Observe that, by definition of $E(w)$ and (A43),

$$\sum_{\substack{w:\mathbf{w}\text{ given} \\ E(w) \geqslant 4m+1}} \exp[-\beta E(w)]$$
$$\leqslant c \exp\{-\tilde{c}\beta[\max(|\mathbf{w}|, 4m+1)]\}$$

where c is independent of β and m.

Using this observation, (A51) and a standard Peierls argument, one gets, for β large,

$$\sum_{\substack{w:\mathbf{w} \cap B \neq \varnothing \\ E(w) \geqslant 4m+1}} \exp[-\beta \tilde{E}(w)]$$
$$\leqslant |B| \exp[-(4m+1/2)\beta] = \beta \exp(-\beta/2) \tag{A53}$$

where the last equality uses the definition of B, in particular $|B| = \beta \exp(4m)$.

Inserting (A.53) in (A.52) and using $\exp(x) - 1 \cong x$, for x small, we get

$$(A52) \leqslant \exp(-c\beta \, \|\Gamma_1\|)$$

Combining this last inequality with (A50) and (A48), we obtain (A45), because $\|\Gamma\| = \|\Gamma_1\| + \|\Gamma_2\|$. This concludes the proof of Proposition A2. ∎

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